

Local well-posedness of compressible-incompressible two-phase flows with phase transitions

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Abstract

This paper is concerned with the basic model for compressible and incompressible two phase flows with phase transitions. The flows are separated by nearly flat interface represented as a graph over the $N - 1$ dimensional Euclidean space \mathbb{R}^{N-1} ($N \geq 2$). The local well-posedness is proved by the Banach fixed point theorem based on the maximal L_p - L_q regularity theorem for the linearized problem.

1 Introduction

The three states of matter are solids, liquids and gases. The flow consisting of two phases, which are mixed and interacting each other, is called the two phase flow. To analyze the two phase flow is the important problem in the field of the fluid machine. For example, nowadays it is known that cavitation noise and the damage of hard material for turbo-machines and ship propellers are induced by impulsive pressures that are caused by the collapse of cloud of bubbles in the water. In fact, K. Yamamoto [20] investigated that the water jets injected from submerged nozzle with narrow orifice were observed by a high-speed video camera and he found that the several times rebound of the cloud of bubbles creates very strong pressure pulses which cause the cavitation noise and the damage of hard material. Moreover, D. Rosseinelli et al [9] showed the simulation of cloud cavitation collapse by the high performance computer. Thus, the study of the cavitation, which is described by the compressible and incompressible fluid flow mathematically, has new development in the experimental fluid mechanics and computational fluid mechanics, rather recently. On the other hand, the mathematical approach to two phase problem with liquids and bubbles is rare, even when they are described by the compressible and incompressible viscous fluid flow with sharp interface. The author knows only results due to Denisova [2] and Kubo, Shibata and Soga [4]. In this paper, we start with the modeling of the two phase problem which can be found in the usual engineering text books without any mathematical proof and we prove the local well-posedness in the phase transition case, which has not been yet treated in any mathematical literature as far as the author knows.

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2 Modeling.

Following the Prüss idea in [6], we discuss the modeling. Let Ω be a domain in the N dimensional Euclidean space \mathbb{R}^N ($N \geq 2$) with boundary Γ_0 . Let Ω_- be a subdomain of Ω with boundary Γ . We assume that $\Gamma = \partial\Omega_- \subset \Omega$ and that $\Gamma_0 \cap \Gamma = \emptyset$. Set $\Omega_+ = \Omega - \overline{\Omega_-}$. Let $\varphi = \varphi(\xi, t) = (\varphi_1(\xi, t), \dots, \varphi_N(\xi, t))$ be a function defined on the closure of Ω for each time variable $t \in (0, T)$, $\xi = (\xi_1, \dots, \xi_N)$ being the reference coordinate system. We assume that the map $\xi \rightarrow \varphi(\xi, t)$ is one to one for each $t \in (0, T)$ [†]. Set $(\partial_t \varphi)(\xi, t) = \mathbf{v}(x, t)$ with $x = \varphi(\xi, t)$,

$$\Omega_{\pm}(t) = \{x = \varphi(\xi, t) \mid \xi \in \Omega_{\pm}\}, \quad \Gamma(t) = \{x = \varphi(\xi, t) \mid \xi \in \Gamma\},$$

and $\dot{\Omega}(t) = \Omega_-(t) \cup \Omega_+(t)$. Let $\mathbf{n}_{\Gamma(t)}$ be the unit outer normal to $\Gamma(t)$ pointed from $\Omega_-(t)$ to $\Omega_+(t)$ and let \mathbf{n}_{Γ_0} the unit outer normal to Γ_0 . Set

$$[[v]] = v_- - v_+ \quad (\text{the jump of } v \text{ accross } \Gamma(t))$$

for any v defined on $\dot{\Omega}(t)$. Here and hereafter, we write $v_{\pm} = v|_{\Omega_{\pm}(t)}$. Moreover, given v_{\pm} defined on $\Omega_{\pm}(t)$, we define v by $v(x) = v_{\pm}(x)$ for $x \in \Omega_{\pm}(t)$. Let $H_{\Gamma} = -\text{div}_{\Gamma} \mathbf{n}_{\Gamma}$ be the mean curvature of $\Gamma(t)$. During our modeling, we use the well-known Reynolds transport theorem:

$$\frac{d}{dt} \int_{\dot{\Omega}(t)} f \, dx = \int_{\dot{\Omega}(t)} \partial_t f \, dx + \int_{\Gamma(t)} [[f]] \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} f \mathbf{v} \cdot \mathbf{n}_{\Gamma_0} \, d\nu,$$

where $d\nu$ represents the surface element not only of $\Gamma(t)$ but also of Γ_0 . In this orientation, we know that

$$\frac{d}{dt} |\Gamma(t)| = - \int_{\Gamma(t)} H_{\Gamma} \mathbf{v} \cdot \mathbf{n}_{\Gamma} \, d\nu \quad (2.1)$$

Here and in the following, we use bold small letters to denote N -vector and N -vector valued functions and the bold capital letters to denote $N \times N$ matrix and $N \times N$ matrix valued functions, respectively. For $\mathbf{v} = (v_1, \dots, v_N)$ and $\mathbf{w} = (w_1, \dots, w_N)$, we set $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = \sum_{j=1}^N v_j w_j$, which is the usual inner product of \mathbb{R}^N .

In the following, we use the following notation:

- $\rho : \dot{\Omega}(t) \rightarrow \mathbb{R}_+$ is the mass field,
- $\mathbf{u} : \dot{\Omega}(t) \rightarrow \mathbb{R}^N$ the velocity field,
- $\pi : \dot{\Omega}(t) \rightarrow \mathbb{R}$ the pressure field,
- $\mathbf{T} : \dot{\Omega}(t) \rightarrow \{\mathbf{A} \in GL_N(\mathbb{R}) \mid {}^T \mathbf{A} = \mathbf{A}\}$ the stress tensor field,
- $\theta : \dot{\Omega}(t) \rightarrow \mathbb{R}_+$ the thermal field,
- $e : \dot{\Omega}(t) \rightarrow \mathbb{R}$ the internal energy,
- $\mathbf{q} : \dot{\Omega}(t) \rightarrow \mathbb{R}^N$ the heat flux,

[†]Since this subsection is concerned with the modeling, we do not care the regularity of boundary and the map φ . Moreover, we do not mention any integrability of functions regorously. These are formulated mathematically in sections 2.

- $\eta : \dot{\Omega}(t) \rightarrow \mathbb{R}$ the entropy,

where, we have set $\mathbb{R}_+ = (0, \infty)$. For the modeling, we use the following Navier-Stokes-Fourier system of equations: for $x \in \dot{\Omega}(t)$ and $t > 0$

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0; \quad (2.2)$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div} \mathbf{T} = 0; \quad (2.3)$$

$$\partial_t\left(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\right) + \operatorname{div}\left(\left(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e\right)\mathbf{u}\right) - \operatorname{div}(\mathbf{T}\mathbf{u} - \mathbf{q}) = 0. \quad (2.4)$$

Here, for any $\mathbf{u} = (u_1, \dots, u_N)$, $\mathbf{u} \otimes \mathbf{u}$ is the $N \times N$ matrix whose (i, j) component is $u_i u_j$, and for any $\mathbf{w} = (w_1, \dots, w_N)$ and $\mathbf{S} = (S_{ij})$ the divergence forms $\operatorname{div} \mathbf{w}$ and $\operatorname{div} \mathbf{S}$ are defined by

$$\operatorname{div} \mathbf{w} = \sum_{j=1}^N \partial_j w_j, \quad \operatorname{div} \mathbf{S} = \left(\sum_{j=1}^N \partial_j S_{1j}, \dots, \sum_{j=1}^N \partial_j S_{Nj}\right).$$

During our discussion of the jump condition on $\Gamma(t)$ and boundary condition on Γ_0 , we assume that $\mathbf{v} \neq \mathbf{u}$ on $\Gamma(t)$, but $\mathbf{v} = \mathbf{u}$ on Γ_0 .

First, we consider the mass conservation:

$$\frac{d}{dt} \int_{\dot{\Omega}(t)} \rho \, dx = 0. \quad (2.5)$$

By (2.2) and the Reynolds transport theorem, we have

$$\begin{aligned} \frac{d}{dt} \int_{\dot{\Omega}(t)} \rho \, dx &= \int_{\dot{\Omega}(t)} \partial_t \rho \, dx + \int_{\Gamma(t)} [[\rho]] \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} \rho \mathbf{u} \cdot \mathbf{n}_{\Gamma_0} \, d\nu \\ &= - \int_{\dot{\Omega}(t)} \operatorname{div}(\rho \mathbf{u}) \, dx + \int_{\Gamma(t)} [[\rho]] \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} \rho \mathbf{u} \cdot \mathbf{n}_{\Gamma_0} \, d\nu \\ &= - \int_{\Gamma(t)} [[\rho(\mathbf{u} - \mathbf{v})]] \cdot \mathbf{n}_{\Gamma(t)} \, d\nu. \end{aligned}$$

Thus, to obtain (4.2), it is sufficient to assume that

$$[[\rho(\mathbf{u} - \mathbf{v})]] \cdot \mathbf{n}_{\Gamma(t)} = 0 \quad \text{on } \Gamma(t). \quad (2.6)$$

In this case, $\rho_+(\mathbf{u}_+ - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} = \rho_-(\mathbf{u}_- - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)}$ on $\Gamma(t)$, so that the phase flux j is defined by

$$j = \rho_+(\mathbf{u}_+ - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} = \rho_-(\mathbf{u}_- - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)}. \quad (2.7)$$

- When $j = 0$, we have $[[\mathbf{u}]] \cdot \mathbf{n}_{\Gamma} = 0$.
- When $j \neq 0$ and $[[\rho]] \neq 0$, we have

$$j = \frac{[[\mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}}{[[1/\rho]]}. \quad (2.8)$$

- When $j \neq 0$ and $[[\rho]] = 0$, j can not be decided by the velocity field \mathbf{u} .

The case where $j = 0$ is called without phase transition and the case where $j \neq 0$ with phase transition.

Next, we consider the conservation of momentum:

$$\frac{d}{dt} \int_{\dot{\Omega}(t)} \rho \mathbf{u} \, dx = 0. \quad (2.9)$$

By (2.3) and the Reynolds transport theorem, we have

$$\begin{aligned} \frac{d}{dt} \int_{\dot{\Omega}(t)} \rho \mathbf{u} \, dx &= - \int_{\dot{\Omega}(t)} \operatorname{div} (\rho \mathbf{u} \otimes \mathbf{u}) \, dx + \int_{\dot{\Omega}(t)} \operatorname{div} \mathbf{T} \, dx \\ &= - \int_{\Gamma(t)} ([[\rho \mathbf{u} \otimes (\mathbf{u} - \mathbf{v}) - [[\mathbf{T}]]] \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} \mathbf{T} \mathbf{n}_{\Gamma_0} \, d\nu. \end{aligned}$$

Thus, in order that $\frac{d}{dt} \int_{\dot{\Omega}(t)} \rho \mathbf{u} \, dx = 0$ holds, it is sufficient to assume that

$$\begin{cases} [[\rho \mathbf{u} \otimes (\mathbf{u} - \mathbf{v}) - \mathbf{T}]] \mathbf{n}_{\Gamma(t)} = \operatorname{div}_{\Gamma} \mathbf{T}_{\Gamma} & \text{on } \Gamma(t), \\ \mathbf{T} \mathbf{n}_{\Gamma_0} = 0 & \text{on } \Gamma_0 \end{cases} \quad (2.10)$$

Here, \mathbf{T}_{Γ} is the stress tensor field on $\Gamma(t)$. Note that $\int_{\Gamma(t)} \operatorname{div}_{\Gamma} \mathbf{T}_{\Gamma} \, d\nu = 0$. We assume that $\operatorname{div}_{\Gamma} \mathbf{T}_{\Gamma} = -\sigma H_{\Gamma} \mathbf{n}_{\Gamma(t)}$, where σ is a non-negative constant describing the coefficient of surface tension.

We represent the interface condition (2.10) with the help of the phase flux j as follows:

$$[[\rho \mathbf{u} \otimes (\mathbf{u} - \mathbf{v})]] \cdot \mathbf{n}_{\Gamma(t)} = \rho_- \mathbf{u}_- (\mathbf{u}_- - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} - \rho_+ \mathbf{u}_+ (\mathbf{u}_+ - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} = j[[\mathbf{u}]].$$

Moreover, by (2.2) we rewrite (2.3) as follows:

$$\partial_t (\rho \mathbf{u}) + \operatorname{div} (\rho \mathbf{u} \otimes \mathbf{u}) = \mathbf{u} (\partial_t \rho + \operatorname{div} (\rho \mathbf{u})) + \rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}).$$

Summing up, we have obtained

$$\begin{cases} \rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \operatorname{div} \mathbf{T} = 0 & \text{in } \dot{\Omega}(t), \\ j[[\mathbf{u}]] - [[\mathbf{T} \mathbf{n}_{\Gamma(t)}]] = -\sigma H_{\Gamma} \mathbf{n}_{\Gamma(t)} & \text{on } \Gamma(t), \\ \mathbf{T} \mathbf{n}_{\Gamma_0} = 0 & \text{on } \Gamma_0. \end{cases} \quad (2.11)$$

Here and in the following, for any N -vector functions $\mathbf{w} = (w_1, \dots, w_N)$, $\mathbf{z} = (z_1, \dots, z_N)$ and scalar function f , we set $\mathbf{w} \cdot \nabla f = \sum_{j=1}^N w_j \partial_j f$, and $\mathbf{w} \cdot \nabla \mathbf{z}$ is an N -vector function whose i th component is $\mathbf{w} \cdot \nabla z_i$.

Next, we consider the balance of energy. We look for a sufficient condition to obtain the conservation of energy:

$$\frac{d}{dt} \left(\int_{\dot{\Omega}(t)} \left(\frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) dx + \sigma |\Gamma(t)| \right) = 0. \quad (2.12)$$

By (2.4) and the Reynolds transport theorem, we have

$$\begin{aligned} \frac{d}{dt} \int_{\dot{\Omega}(t)} \left(\frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) dx \\ = - \int_{\Gamma(t)} \left[\left(\frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) (\mathbf{u} - \mathbf{v}) - (\mathbf{T} \mathbf{u} - \mathbf{q}) \right] \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} (\mathbf{T} \mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_{\Gamma_0} \, d\nu, \end{aligned}$$

which, combined with (2.1), furnishes that

$$\begin{aligned} \frac{d}{dt} \left(\int_{\hat{\Omega}(t)} \left(\frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) dx + \sigma |\Gamma(t)| \right) &= \int_{\Gamma_0} (\mathbf{T}\mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_{\Gamma_0} d\nu \\ &\quad - \int_{\Gamma(t)} \left(\left[\left(\frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) (\mathbf{u} - \mathbf{v}) - (\mathbf{T}\mathbf{u} - \mathbf{q}) \right] \cdot \mathbf{n}_{\Gamma(t)} + \sigma H_{\Gamma} \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \right) d\nu. \end{aligned}$$

Thus, in order to obtain (2.12), it is sufficient to assume that

$$\begin{cases} \left[\left(\frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) (\mathbf{u} - \mathbf{v}) - (\mathbf{T}\mathbf{u} - \mathbf{q}) \right] \cdot \mathbf{n}_{\Gamma(t)} + \sigma H_{\Gamma} \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = 0 & \text{on } \Gamma(t), \\ (\mathbf{T}\mathbf{u} - \mathbf{q}) \cdot \mathbf{n}_{\Gamma_0} = 0 & \text{on } \Gamma_0. \end{cases} \quad (2.13)$$

Since $\mathbf{T}\mathbf{n}_{\Gamma_0} = 0$ on Γ_0 , we assume that $\mathbf{q} \cdot \mathbf{n}_{\Gamma_0} = 0$ on Γ_0 . By (2.7) and (2.11),

$$\begin{aligned} &\left[\left(\frac{\rho}{2} |\mathbf{u}|^2 (\mathbf{u} - \mathbf{v}) \right) \right] \cdot \mathbf{n}_{\Gamma(t)} \\ &= \frac{1}{2} (|\mathbf{u}_- - \mathbf{v} + \mathbf{v}|^2 - |\mathbf{u}_+ - \mathbf{v} + \mathbf{v}|^2) = \frac{1}{2} [|\mathbf{u} - \mathbf{v}|^2] + \mathbf{J}[[\mathbf{u}]] \cdot \mathbf{v} \\ &= \frac{1}{2} [|\mathbf{u} - \mathbf{v}|^2] + [[\mathbf{T}\mathbf{n}_{\Gamma(t)}]] \cdot \mathbf{v} - \sigma H_{\Gamma} \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \end{aligned}$$

Since $[\rho e(\mathbf{u} - \mathbf{v})] \cdot \mathbf{n}_{\Gamma(t)} = \mathbf{J}[[e]]$, the first equation of (2.13) becomes:

$$\frac{1}{2} [|\mathbf{u} - \mathbf{v}|^2] + \mathbf{J}[[e]] - [[\mathbf{T}(\mathbf{u} - \mathbf{v})]] \cdot \mathbf{n}_{\Gamma(t)} + [[\mathbf{q}]] \cdot \mathbf{n}_{\Gamma(t)} = 0.$$

Moreover, using (2.2) and (2.3), we rewrite (2.4) as follows:

$$\begin{aligned} &\partial_t \left(\frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) + \operatorname{div} \left(\left(\frac{\rho}{2} |\mathbf{u}|^2 + \rho e \right) \mathbf{u} \right) - \operatorname{div} (\mathbf{T}\mathbf{u} - \mathbf{q}) \\ &= \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) \partial_t \rho + \rho (\mathbf{u} \cdot \partial_t \mathbf{u} + \partial_t e) + \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) \operatorname{div} (\rho \mathbf{u}) + \rho \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u} + \nabla e) \\ &\quad - (\operatorname{div} \mathbf{T}) \cdot \mathbf{u} - \mathbf{T} : \nabla \mathbf{u} + \operatorname{div} \mathbf{q} \\ &= \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) (\partial_t \rho + \operatorname{div} (\rho \mathbf{u})) + \mathbf{u} \cdot (\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \operatorname{div} \mathbf{T}) \\ &\quad + \rho (\partial_t e + \mathbf{u} \cdot \nabla \mathbf{u}) - \mathbf{T} : \nabla \mathbf{u} + \operatorname{div} \mathbf{q}. \end{aligned}$$

Here, we have set $\mathbf{T} : \nabla \mathbf{u} = \sum_{i,j=1}^N T_{ij} \partial_i u_j$. Thus, we have

$$\rho (\partial_t e + \mathbf{u} \cdot \nabla e) + \operatorname{div} \mathbf{q} - \mathbf{T} : \nabla \mathbf{u} = 0.$$

Summing up, we have obtained

$$\begin{cases} \rho (\partial_t e + \mathbf{u} \cdot \nabla e) + \operatorname{div} \mathbf{q} - \mathbf{T} : \nabla \mathbf{u} = 0 & \text{in } \hat{\Omega}(t), \\ \frac{1}{2} [|\mathbf{u} - \mathbf{v}|^2] + \mathbf{J}[[e]] - [[\mathbf{T}(\mathbf{u} - \mathbf{v})]] \cdot \mathbf{n}_{\Gamma(t)} + [[\mathbf{q}]] \cdot \mathbf{n}_{\Gamma(t)} = 0 & \text{on } \Gamma(t), \\ \mathbf{q} \cdot \mathbf{n}_{\Gamma_0} = 0 & \text{on } \Gamma_0. \end{cases} \quad (2.14)$$

The number of interface conditions is so far not enough. To find one more condition, we consider the law of entropy increase:

$$\frac{d}{dt} \int_{\hat{\Omega}(t)} \rho \eta dx \geq 0. \quad (2.15)$$

For this purpose, we introduce the constitutive laws in the phases. According to the Newton law, the stress tensor \mathbf{T} is given by

$$\mathbf{T} = 2\mu\mathbf{D}(\mathbf{u}) + (\lambda - \mu)\text{div } \mathbf{u}\mathbf{I} - \pi\mathbf{I}.$$

Here, $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the strain tensor field, \mathbf{I} the $N \times N$ identity matrix, μ and λ are the first and second viscosity coefficients satisfying the condition:

$$\mu > 0, \quad \lambda \geq \frac{N-2}{N}\mu. \quad (2.16)$$

To prove local well-posedness, it suffices to assume that $\mu > 0$ and $\lambda > 0$. According to the Fourier law, the heat flux \mathbf{q} is given by

$$\mathbf{q} = -d\nabla\theta. \quad (2.17)$$

with thermal conductivity d satisfying the condition: $d > 0$. Moreover, the first law of thermodynamics tells us that the internal energy e , the entropy η , and the pressure term π have the relation:

$$de = \theta d\eta + \frac{\pi}{\rho^2} d\rho. \quad (2.18)$$

We define the free energy ψ for the unit mass and the specific heat κ by

$$\psi = e - \theta\eta, \quad \kappa = \frac{\partial e}{\partial \theta}, \quad (2.19)$$

respectively. We assume that $\kappa > 0$. Since $\frac{\partial e}{\partial \eta} = \theta$ and $\frac{\partial e}{\partial \rho} = \frac{\pi}{\rho^2}$ as follows from (2.18), by (2.2)

$$\begin{aligned} \partial_t e + \mathbf{u} \cdot \nabla e &= \frac{\partial e}{\partial \eta}(\partial_t \eta + \mathbf{u} \cdot \nabla \eta) + \frac{\partial e}{\partial \rho}(\partial_t \rho + \mathbf{u} \cdot \nabla \rho) \\ &= \theta(\partial_t \eta + \mathbf{u} \cdot \nabla \eta) - \frac{\pi}{\rho} \text{div } \mathbf{u}. \end{aligned} \quad (2.20)$$

In addition,

$$\mathbf{T} : \nabla \mathbf{u} = 2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu)(\text{div } \mathbf{u})^2 - \pi \text{div } \mathbf{u}, \quad (2.21)$$

which, combined with the first equation of (2.14), (2.20), and (2.21), furnishes

$$\rho\theta(\partial_t \eta + \mathbf{u} \cdot \nabla \eta) - \text{div } (d\nabla\theta) - (2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu)(\text{div } \mathbf{u})^2) = 0. \quad (2.22)$$

On the other hand, we have

$$\partial_t(\rho\eta) + \text{div } (\rho\eta\mathbf{u}) = \eta(\partial_t \rho + \text{div } (\rho\mathbf{u})) + \rho(\partial_t \eta + \mathbf{u} \cdot \nabla \eta) = \rho(\partial_t \eta + \mathbf{u} \cdot \nabla \eta). \quad (2.23)$$

In the following, we assume that

$$\theta > 0, \quad [[\theta]] = 0. \quad (2.24)$$

Since θ represents the absolute temperature, $\theta > 0$ is natural assumption. While phase transition happens, the temperature does not change, so that $[[\theta]] = 0$ is also natural assumption. By (2.22) and (2.23)

$$\partial_t(\rho\eta) + \text{div } (\rho\eta\mathbf{u}) = \frac{1}{\theta} \{ \text{div } (d\nabla\theta) + 2\mu|\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu)(\text{div } \mathbf{u})^2 \}. \quad (2.25)$$

By the Reynolds transport theorem, (2.25), and the divergence theorem of Gauss,

$$\begin{aligned}
& \frac{d}{dt} \int_{\dot{\Omega}(t)} \rho \eta \, dx \\
&= - \int_{\dot{\Omega}(t)} \operatorname{div}(\rho \eta \mathbf{u}) \, dx + \int_{\dot{\Omega}(t)} \frac{1}{\theta} \{ \operatorname{div}(d\nabla \theta) + (2\mu |\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu)(\operatorname{div} \mathbf{u})^2) \} \, dx \\
&\quad + \int_{\Gamma(t)} [[\rho \eta]] \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma_0} \rho \eta \mathbf{u} \cdot \mathbf{n}_{\Gamma_0} \, d\nu \\
&= - \int_{\Gamma(t)} [(\rho \eta)(\mathbf{u} - \mathbf{v})] \cdot \mathbf{n}_{\Gamma(t)} \, d\nu + \int_{\Gamma(t)} \left[\left[\frac{d}{\theta} \nabla \theta \right] \cdot \mathbf{n}_{\Gamma(t)} \right] \, d\nu + \int_{\Gamma_0} \frac{1}{\theta} (d\nabla \theta \cdot \mathbf{n}_{\Gamma_0}) \, d\nu \\
&\quad + \int_{\dot{\Omega}(t)} \left\{ \frac{d|\nabla \theta|^2}{\theta^2} + 2\mu |\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu)(\operatorname{div} \mathbf{u})^2 \right\} \, dx.
\end{aligned}$$

Since $2\mu |\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu)(\operatorname{div} \mathbf{u})^2 \geq 0$ as follows from (2.16), to obtain (2.15) it is sufficient to assume that

$$\begin{aligned}
[[(\rho \eta)(\mathbf{u} - \mathbf{v}) - \frac{d}{\theta} \nabla \theta]] \cdot \mathbf{n}_{\Gamma(t)} &= 0 \quad \text{on } \Gamma(t), \\
d\nabla \theta \cdot \mathbf{n}_{\Gamma_0} &= 0 \quad \text{on } \Gamma_0.
\end{aligned} \tag{2.26}$$

Moreover, by $[[\theta]] = 0$ and (2.7), the first equation of (2.26) becomes

$$j[[\theta \eta]] - [[d\nabla \theta]] \cdot \mathbf{n}_{\Gamma(t)} = 0 \quad \text{on } \Gamma(t), \tag{2.27}$$

which is called the Stefan law. In fact, by (2.7) and $[[\theta]] = 0$,

$$\begin{aligned}
0 &= [(\rho \eta)(\mathbf{u} - \mathbf{v}) - \frac{d}{\theta} \nabla \theta] \cdot \mathbf{n}_{\Gamma(t)} \\
&= (\rho_- \eta_-)(\mathbf{u}_- - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} - (\rho_+ \eta_+)(\mathbf{u}_+ - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)} - \frac{(d_- \nabla \theta_- - d_+ \nabla \theta_+) \cdot \mathbf{n}_{\Gamma(t)}}{\theta} \\
&= \frac{1}{\theta} (j[[\theta \eta]] - [[d\nabla \theta]] \cdot \mathbf{n}_{\Gamma(t)}).
\end{aligned}$$

Note that the Stefan law becomes the usual jump condition: $[[d\nabla \theta]] \cdot \mathbf{n}_{\Gamma(t)} = 0$ on $\Gamma(t)$ provided that $j = 0$.

Next, assuming that $j \neq 0$ and $[[\rho]] \neq 0$ and using (2.27), we rewrite (2.14). Given \mathbf{w} , we set $\mathcal{T}_{\mathbf{n}_{\Gamma(t)}} \mathbf{w} = \mathbf{w} - (\mathbf{w} \cdot \mathbf{n}_{\Gamma(t)}) \mathbf{n}_{\Gamma(t)}$, which is the tangential part of \mathbf{w} along $\mathbf{n}_{\Gamma(t)}$. Since $\mathbf{w} = (\mathbf{w} \cdot \mathbf{n}_{\Gamma(t)}) \mathbf{n}_{\Gamma(t)} + \mathcal{T}_{\mathbf{n}_{\Gamma(t)}} \mathbf{w}$ and $\mathcal{T}_{\mathbf{n}_{\Gamma(t)}} \mathbf{w} \cdot \mathbf{n}_{\Gamma(t)} = 0$, we have

$$|\mathbf{w}|^2 = |\mathbf{w} \cdot \mathbf{n}_{\Gamma(t)}|^2 + |\mathcal{T}_{\mathbf{n}_{\Gamma(t)}} \mathbf{w}|^2. \tag{2.28}$$

In the following, we assume that

$$\mathcal{T}_{\mathbf{n}_{\Gamma(t)}} [[\mathbf{u} - \mathbf{v}]] = 0. \tag{2.29}$$

Especially, we have

$$[[\mathbf{u}]] = ([[\mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}) \mathbf{n}_{\Gamma(t)}. \tag{2.30}$$

Since $|\mathbf{w}|^2 - |\mathbf{v}|^2 = (\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} + \mathbf{v})$ for any \mathbf{w} and \mathbf{v} , by (2.29) $[[|\mathcal{T}_{\mathbf{n}_{\Gamma(t)}}(\mathbf{u} - \mathbf{v})|^2]] = 0$, so that by (2.7) and (2.28)

$$[[\frac{1}{2} |\mathbf{u} - \mathbf{v}|^2]] = \frac{1}{2} [[|\mathbf{u} - \mathbf{v}|^2]] = \frac{j^2}{2} [[\frac{1}{\rho^2}]]. \tag{2.31}$$

Since $e = \psi + \theta\eta$, we have $[[e]] = [[\psi]] + [[\theta\eta]]$. Thus, recalling (2.17) and using (2.27) and (2.31), we rewrite the jump condition in (2.14) as follows:

$$\begin{aligned} 0 &= \frac{j}{2} [[|\mathbf{u} - \mathbf{v}|^2]] + j[[e]] - [[\mathbf{T}(\mathbf{u} - \mathbf{v})]] \cdot \mathbf{n}_{\Gamma(t)} - [[\mathbf{q}]] \cdot \mathbf{n}_{\Gamma(t)} \\ &= j^3 [[\frac{1}{2\rho^2}]] - [[\mathbf{T}(\mathbf{u} - \mathbf{v})]] \cdot \mathbf{n}_{\Gamma(t)} + j[[\psi]]. \end{aligned}$$

Moreover, noting that \mathbf{T}_\pm are symmetric matrices, we have

$$\begin{aligned} [[\mathbf{T}(\mathbf{u} - \mathbf{v})]] \cdot \mathbf{n}_{\Gamma(t)} &= [(\mathbf{u} - \mathbf{v}) \cdot \mathbf{T}\mathbf{n}_{\Gamma(t)}] \\ &= [((\mathbf{u} - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)})\mathbf{n}_{\Gamma(t)} \cdot \mathbf{T}\mathbf{n}_{\Gamma(t)}] + [[\mathcal{T}_{\mathbf{n}_{\Gamma(t)}}(\mathbf{u} - \mathbf{v}) \cdot \mathbf{T}\mathbf{n}_{\Gamma(t)}]]. \end{aligned}$$

By (2.7),

$$[[((\mathbf{u} - \mathbf{v}) \cdot \mathbf{n}_{\Gamma(t)})\mathbf{n}_{\Gamma(t)} \cdot \mathbf{T}\mathbf{n}_{\Gamma(t)}]] = j[[\frac{1}{\rho}\mathbf{n}_{\Gamma(t)} \cdot \mathbf{T}\mathbf{n}_{\Gamma(t)}]].$$

On the other hand, by (2.11) and (2.29) and (2.30),

$$\begin{aligned} [[\mathcal{T}_{\mathbf{n}_{\Gamma(t)}}(\mathbf{u} - \mathbf{v}) \cdot \mathbf{T}\mathbf{n}_{\Gamma(t)}]] &= \mathcal{T}_{\mathbf{n}_{\Gamma(t)}}(\mathbf{u}_- - \mathbf{v}) \cdot [[\mathbf{T}\mathbf{n}_{\Gamma(t)}]] \\ &= \mathcal{T}_{\mathbf{n}_{\Gamma(t)}}(\mathbf{u}_- - \mathbf{v}) \cdot (j[[\mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)} + \sigma H_\Gamma \mathbf{n}_{\Gamma(t)}) = 0. \end{aligned}$$

Thus, we have obtained $0 = j([[\psi]] + j^2[[\frac{1}{2\rho^2}]] - [[\frac{1}{\rho}\mathbf{n}_{\Gamma(t)} \cdot \mathbf{T}\mathbf{n}_{\Gamma(t)}]])$. Since $j \neq 0$, finally we arrive at the condition:

$$[[\psi]] + j^2[[\frac{1}{2\rho^2}]] - [[\frac{1}{\rho}\mathbf{n}_{\Gamma(t)} \cdot \mathbf{T}\mathbf{n}_{\Gamma(t)}]] = 0 \quad \text{on } \Gamma(t), \quad (2.32)$$

which is called the generalized Gibbs-Thomson law.

Finally, we calculate $V_\Gamma := \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)}$. By (2.7) we have $\mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = \mathbf{u}_- \cdot \mathbf{n}_{\Gamma(t)} - \frac{j}{\rho_-}$. When $j = 0$, it follows from (2.7) and (2.29) that $[[\mathbf{u}]] = 0$, so that $\mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = \mathbf{u} \cdot \mathbf{n}_{\Gamma(t)}$. When $j \neq 0$ and $[[\rho]] \neq 0$, by (2.8) we have $j = \frac{[[\mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}}{[[1/\rho]]}$, so that

$$\mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = \mathbf{u}_- \cdot \mathbf{n}_{\Gamma(t)} - \frac{j}{\rho_-} = \frac{[[\rho\mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}}{[[\rho]]}.$$

Summing up, we have obtained

$$\begin{aligned} V_\Gamma &:= \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = \mathbf{u} \cdot \mathbf{n}_{\Gamma(t)} \quad (j = 0), \\ V_\Gamma &:= \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = \frac{[[\rho\mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}}{[[\rho]]} \quad (j \neq 0 \text{ and } [[\rho]] \neq 0). \end{aligned} \quad (2.33)$$

Next, we consider the case where $j \neq 0$ and $[[\rho]] = 0$. In this case, $[[\mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)} = 0$, which, combined with (2.29), furnishes that $[[\mathbf{u}]] = 0$, so that (2.11) becomes

$$[[\mathbf{T}\mathbf{n}_{\Gamma(t)}]] = \sigma H_\Gamma \mathbf{n}_{\Gamma(t)} \quad \text{on } \Gamma(t). \quad (2.34)$$

To derive (2.32), we assume that $[[\rho]] \neq 0$, so that we reconsider the second condition of (2.14). By $[[\mathbf{u}]] = 0$, $[[|\mathbf{u} - \mathbf{v}|^2]] = 0$. By (2.17) and (2.27), $j[[e]] + [[\mathbf{q}]] \cdot \mathbf{n}_{\Gamma(t)} =$

$J[[\psi]]$. In addition, by (2.7), (2.34), and the symmetricity of \mathbf{T}_\pm

$$\begin{aligned} [[\mathbf{T}(\mathbf{u} - \mathbf{v})]] \cdot \mathbf{n}_{\Gamma(t)} &= (\mathbf{u}_- - \mathbf{v}) \cdot \mathbf{T}_- \mathbf{n}_{\Gamma(t)} - (\mathbf{u}_+ - \mathbf{v}) \cdot \mathbf{T}_+ \mathbf{n}_{\Gamma(t)} \\ &= (\mathbf{u}_- - \mathbf{v}) \cdot [[\mathbf{T}\mathbf{n}_{\Gamma(t)}]] = (\mathbf{u}_- - \mathbf{v}) \cdot \sigma H_\Gamma \mathbf{n}_{\Gamma(t)} = J \frac{\sigma}{\rho_-} H_\Gamma. \end{aligned}$$

Since $J \neq 0$, the second equation of (2.14) becomes

$$[[\psi]] - \frac{\sigma}{\rho_-} H_\Gamma = 0 \quad \text{on } \Gamma(t). \quad (2.35)$$

Noting that $\partial_t \rho + \mathbf{u} \cdot \nabla \rho = -\rho \operatorname{div} \mathbf{u}$ as follows from (2.2) and recalling the formulas: $\frac{\partial e}{\partial \theta} = \kappa$ and $\frac{\partial e}{\partial \rho} = \frac{\pi}{\rho^2}$ (cf. (2.18) and (2.19)), we have

$$\begin{aligned} \rho(\partial_t e + \mathbf{u} \cdot \nabla e) &= \rho \left(\frac{\partial e}{\partial \theta} \partial_t \theta + \frac{\partial e}{\partial \rho} \partial_t \rho + \frac{\partial e}{\partial \theta} \mathbf{u} \cdot \nabla \theta + \frac{\partial e}{\partial \rho} \mathbf{u} \cdot \nabla \rho \right) \\ &= \rho \kappa (\partial_t \theta + \mathbf{u} \cdot \nabla \theta) - \frac{\pi}{\rho} \operatorname{div} \mathbf{u}, \end{aligned}$$

which, combined with the first equation in (2.14) and (2.21), furnishes that

$$\rho \kappa (\partial_t \theta + \mathbf{u} \cdot \nabla \theta) - \operatorname{div} (d \nabla \theta) - (2\mu |\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu) (\operatorname{div} \mathbf{u})^2) + \pi \left(1 - \frac{1}{\rho}\right) \operatorname{div} \mathbf{u} = 0.$$

Summing up, we have obtained the equations: for $x \in \dot{\Omega}(t)$ and $t > 0$

$$\begin{aligned} \partial_t \rho + \operatorname{div} (\rho \mathbf{u}) &= 0, \\ \rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \operatorname{div} \mathbf{T} &= 0, \\ \rho \kappa (\partial_t \theta + \mathbf{u} \cdot \nabla \theta) - \operatorname{div} (d \nabla \theta) &= (2\mu |\mathbf{D}(\mathbf{u})|^2 + (\lambda - \mu) (\operatorname{div} \mathbf{u})^2) - \pi \left(1 - \frac{1}{\rho}\right) \operatorname{div} \mathbf{u}, \end{aligned} \quad (2.36)$$

subject to the boundary condition: for $x \in \Gamma_0$ and $t > 0$:

$$\mathbf{T} \mathbf{n}_{\Gamma_0} = 0, \quad d \nabla \theta \cdot \mathbf{n}_{\Gamma_0} = 0 \quad \text{on } \Gamma_0, \quad (2.37)$$

and one of the following interface conditions: for $x \in \Gamma(t)$ and $t > 0$

(1) When $J = 0$,

$$\begin{aligned} [[\mathbf{u}]] &= 0, \quad [[\mathbf{T} \mathbf{n}_{\Gamma(t)}]] = \sigma H_\Gamma \mathbf{n}_{\Gamma(t)}, \quad [[\theta]] = 0, \\ [[d \nabla \theta \cdot \mathbf{n}_{\Gamma(t)}]] &= 0, \quad \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = \mathbf{u} \cdot \mathbf{n}_{\Gamma(t)}. \end{aligned} \quad (2.38)$$

(2) When $J \neq 0$ and $[[\rho]] \neq 0$,

$$\begin{aligned} \mathcal{T}_{\mathbf{n}_{\Gamma(t)}} [[\mathbf{u}]] &= 0, \quad J [[\mathbf{u}]] - [[\mathbf{T} \mathbf{n}_{\Gamma(t)}]] = -\sigma H_\Gamma \mathbf{n}_{\Gamma(t)}, \quad [[\theta]] = 0, \\ J [[\theta \eta]] - [[d \nabla \theta \cdot \mathbf{n}_{\Gamma(t)}]] &= 0, \quad [[\psi]] + J^2 \left[\frac{1}{2\rho^2} \right] - \left[\left[\frac{1}{\rho} \mathbf{n}_{\Gamma(t)} \mathbf{T} \mathbf{n}_{\Gamma(t)} \right] \right] = 0, \\ \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} &= \frac{[[\rho \mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}}{[[\rho]]}, \quad J = \frac{[[\mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}}{[[1/\rho]]}. \end{aligned} \quad (2.39)$$

(3) When $J \neq 0$ and $\rho = \rho_- = \rho_+$ (constants),

$$\begin{aligned} [[\mathbf{u}]] &= 0, \quad [[\mathbf{T} \mathbf{n}_\Gamma]] = \sigma H_\Gamma \mathbf{n}_{\Gamma(t)}, \quad [[\theta]] = 0, \quad J [[\theta \eta]] - [[d \nabla \theta \cdot \mathbf{n}_{\Gamma(t)}]] = 0, \\ \rho [[\psi]] - \sigma H_\Gamma &= 0, \quad \mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = \mathbf{u} \cdot \mathbf{n}_{\Gamma(t)} - \frac{J}{\rho}. \end{aligned} \quad (2.40)$$

Remark 1. Assuming that $\Omega_- = \Omega$ and $\Omega_+ = \emptyset$, we have the one phase problem. In this case, as boundary condition on Γ_0 , we have

$$\mathbf{Tn}_{\Gamma_0} = \sigma H_{\Gamma_0} \mathbf{n}_{\Gamma(t)}, \quad d\nabla\theta \cdot \mathbf{n}_{\Gamma_0} = 0 \quad \text{on } \Gamma_0.$$

3 Problem

The problem of this paper is concerned with the compressible and incompressible two phase flow separated by a nearly flat interface with phase transition. Let $h_0(x')$ be a given function with respect to $x' = (x_1, \dots, x_{N-1})$ and we set

$$\begin{aligned} \Omega_{\pm} &= \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid \pm(x_N - h_0(x')) > 0 \text{ for } x' \in \mathbb{R}^{N-1}\}, \\ \Gamma &= \{x \in \mathbb{R}^N \mid x_N = h(x') \text{ for } x' \in \mathbb{R}^{N-1}\}. \end{aligned}$$

In this case, $\Omega = \mathbb{R}^N$ and $\Gamma_0 = \emptyset$. Let $h(x', t)$ be a unknown function and we assume that the time evolutions of domains Ω_{\pm} and the surface Γ are given by

$$\begin{aligned} \Omega_{\pm}(t) &= \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid \pm(x_N - h(x', t)) > 0 \text{ for } x' \in \mathbb{R}^{N-1}\}, \\ \Gamma(t) &= \{x \in \mathbb{R}^N \mid x_N = h(x', t) \text{ for } x' \in \mathbb{R}^{N-1}\}. \end{aligned} \quad (3.1)$$

In this case, $\mathbf{n}_{\Gamma(t)} = (-\nabla' h, 1)/\sqrt{1 + |\nabla' h|^2}$, with $\nabla' h = (\partial_1 h, \dots, \partial_{N-1} h)$ ($\partial_j = \partial/\partial x_j$). Moreover, $\varphi(x, t) = (x', x_N + h(x', t))$, so that $\mathbf{v} \cdot \mathbf{n}_{\Gamma(t)} = \partial_t \varphi \cdot \mathbf{n}_{\Gamma(t)} = \partial_t h / \sqrt{1 + |\nabla' h|^2}$.

In view of (2.21), (2.36), (2.37) and (2.39), our problem is given as follows:

For $x \in \Omega_+(t)$, $t > 0$,

$$\begin{cases} \rho_+(\partial_t \mathbf{u}_+ + \mathbf{u}_+ \cdot \nabla \mathbf{u}_+) - \text{Div } \mathbf{T}_+ = 0, & \partial_t \rho_+ + \text{div}(\rho_+ \mathbf{u}_+) = 0, \\ \rho_+ \kappa_+(\partial_t \theta_+ + \mathbf{u}_+ \cdot \nabla \theta_+) - \text{div}(d_+ \nabla \theta_+) - \mathbf{T}_+ : \nabla \mathbf{u}_+ - \frac{\pi}{\rho} \text{div } \mathbf{u}_+ = 0 \end{cases} \quad (3.2)$$

and, for $x \in \Omega_-(t)$, $t > 0$,

$$\begin{cases} \rho_{*-}(\partial_t \mathbf{u}_{*-} + \mathbf{u}_{*-} \cdot \nabla \mathbf{u}_{*-}) - \text{Div } \mathbf{T}_{*-} = 0, & \text{div } \mathbf{u}_{*-} = 0, \\ \rho_{*-} \kappa_{*-}(\partial_t \theta_{*-} + \mathbf{u}_{*-} \cdot \nabla \theta_{*-}) - \text{div}(d_{*-} \nabla \theta_{*-}) - \mathbf{T}_{*-} : \nabla \mathbf{u}_{*-} = 0 \end{cases}$$

subject to the jump conditions: for $x \in \Gamma(t)$ and $t > 0$,

$$\begin{cases} \left[\left[\frac{1}{\rho} \right] \right]^2 \mathbf{n}_{\Gamma} - [[\mathbf{Tn}_{\Gamma}]] = -\sigma H_{\Gamma} \mathbf{n}_{\Gamma}, & \mathcal{T}_{\mathbf{n}_{\Gamma(t)}} [[\mathbf{u}]] = 0, \\ \mathbf{j} [[\theta \eta]] - \left[\left[d \frac{\partial \theta}{\partial \mathbf{n}_{\Gamma}} \right] \right] = 0, & [[\theta]] = 0, \\ \left[[\psi] \right] + \left[\left[\frac{1}{2\rho^2} \right] \right]^2 - \left[\left[\frac{1}{\rho} \mathbf{n}_{\Gamma} \cdot \mathbf{Tn}_{\Gamma} \right] \right] = 0, & \partial_t h = \frac{[[\rho \mathbf{u}]] \cdot (-\nabla' h, 1)}{[[\rho]]}, \\ \mathbf{j} = \frac{[[\rho \mathbf{u}]] \cdot \mathbf{n}_{\Gamma(t)}}{[[\rho]]}, \end{cases} \quad (3.3)$$

and the initial conditions:

$$\begin{aligned} (\rho_+, \mathbf{u}_+, \theta_+) |_{t=0} &= (\rho_{*+} + \rho_{0+}, \mathbf{u}_{0+}, \theta_* + \theta_{0+}) \text{ in } \Omega_+, \\ (\mathbf{u}_-, \theta_-) |_{t=0} &= (\mathbf{u}_{0+}, \theta_* + \theta_{0+}) \text{ in } \Omega_-, \quad h|_{t=0} = h_0 \text{ on } \Gamma. \end{aligned} \quad (3.4)$$

Here, $\rho_{*\pm}$, θ_* and σ are positive constants describing the reference mass densities of Ω_\pm , the reference temperature of Ω_\pm and the coefficient of the surface tension, respectively. Moreover, $\mathbf{T}_\pm = \mathbf{S}_\pm - \pi_\pm \mathbf{I}$ with

$$\begin{aligned}\mathbf{S}_+ &= \mathbf{S}_+(\mathbf{u}_+, \rho_+, \theta_+) = \mu_+ \mathbf{D}(\mathbf{u}_+) + (\lambda_+ - \mu_+) \operatorname{div} \mathbf{u} \mathbf{I}, \\ \mathbf{S}_- &= \mathbf{S}_-(\mathbf{u}_-, \theta_-) = \mu_- \mathbf{D}(\mathbf{u}_-).\end{aligned}$$

Here, $d_+ = d_+(\rho, \theta)$, $\mu_+ = \mu_+(\rho, \theta)$, $\lambda_+ = \lambda_+(\rho, \theta)$, $\kappa_+ = \kappa_+(\rho, \theta)$ are positive C^∞ functions with respect to $(\rho, \theta) \in (0, \infty) \times (0, \infty)$, and $\psi_+(\theta, \rho)$ and $\eta_+(\theta, \rho)$ are real valued C^∞ functions with respect to $(\rho, \theta) \in (0, \infty) \times (0, \infty)$, while $d_- = d_-(\theta)$, $\mu_- = \mu_-(\theta)$, $\kappa_- = \kappa_-(\theta)$ are positive C^∞ functions with respect to $\theta \in (0, \infty)$, and $\psi_-(\theta)$ and $\eta_-(\theta)$ are real valued C^∞ functions with respect to $\theta \in (0, \infty)$. Moreover, we also assume that π_+ is given by $\pi_+ = P_+(\rho, \theta)$, where P_+ is some C^∞ function with respect to $(\rho, \theta) \in (0, \infty) \times (0, \infty)$ such that $\frac{\partial P_+}{\partial \rho} > 0$ for any $(\rho, \theta) \in (0, \infty) \times (0, \infty)$.

The main purpose of this paper is to show the local wellposedness of problem (3.2), (3.3) and (3.4) in the maximal L_p - L_q regularity class under the assumption that $\rho_{*\pm}$ and θ_* satisfy the condition:

$$\rho_{*-} \neq \rho_{*+}, \quad \psi_-(\theta_*) - \psi_+(\rho_{*+}, \theta_*) + \left(\frac{1}{\rho_{*-}} - \frac{1}{\rho_{*+}} \right) P_+(\rho_{*+}, \theta_*) = 0. \quad (3.5)$$

To state our main result, we transform $\Gamma(t)$ to the flat interface. Set

$$\mathbb{R}_\pm^N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid \pm x_N > 0\}, \quad \mathbb{R}_0^N = \{x \in \mathbb{R}^N \mid x_N = 0\}.$$

We transfer the problem given in domains $\Omega_\pm(t)$ to that in $\dot{\mathbb{R}}^N = \mathbb{R}_+^N \cup \mathbb{R}_-^N$ with interface \mathbb{R}_0^N . Let $h(x', t)$ be a function appearing in the definition of $\Gamma(t)$ in (3.1). Let $H(x, t)$ be a solution to the equations: $(1 - \Delta)H = 0$ in \mathbb{R}_\pm^N with $H|_{x_N=0} = h(x', t)$, where $\Delta H = \sum_{j=1}^N \partial_j^2 H$. To prove the local well-posedness, we assume that h_0 is small enough, so that we may assume that

$$1 + \frac{\partial}{\partial x_N} H(x, t) \geq \frac{1}{2} \quad \text{for any } x \in \mathbb{R}_\pm^N \text{ and } t \in (0, T). \quad (3.6)$$

If we consider the transformation:

$$y_N = x_N + H(x, t), \quad y_j = x_j \quad (j = 1, \dots, N-1), \quad (3.7)$$

then by (3.6) $\Omega_\pm(t)$ and $\Gamma(t)$ are transformed to \mathbb{R}_\pm^N and \mathbb{R}_0^N , respectively, because $y_N = h(y', t)$ when $x_N = 0$ and $\frac{\partial y_N}{\partial x_N} = 1 + (\frac{\partial H}{\partial x_N})(x, t) \geq \frac{1}{2}$.

Let \mathbf{u}_\pm , ρ_+ , π_- and θ_\pm satisfy problem (3.2), (3.3) and (3.4). Set

$$\begin{aligned}\hat{\mathbf{u}}_\pm(x, t) &= \mathbf{u}_\pm(x', x_N + H(x, t), t), \quad \hat{\rho}_+(x, t) = \rho_+(x', x_N + H(x, t), t), \\ \hat{\pi}_-(x, t) &= \pi_-(x', x_N + H(x, t), t) - P_+(\rho_{*+}, \theta_*), \\ \hat{\theta}_\pm(x, t) &= \theta_\pm(x', x_N + H(x, t), t), \quad \mu_{*+} = \mu_+(\rho_{*+}, \theta_*), \quad \lambda_{*+} = \lambda_+(\rho_{*+}, \theta_*), \\ \kappa_{*+} &= \kappa_+(\rho_{*+}, \theta_*), \quad d_{*+} = d_+(\rho_{*+}, \theta_*), \quad \mu_{*-} = \mu_-(\theta_*), \quad \kappa_{*-} = \kappa_-(\theta_*), \\ d_{*-} &= d_-(\theta_*), \quad \hat{\mu}_+ = \mu_+(\hat{\rho}_+, \hat{\theta}_+), \quad \hat{\lambda}_+ = \lambda_+(\hat{\rho}_+, \hat{\theta}_+), \quad \hat{\mu}_- = \mu_-(\hat{\theta}_-), \\ \hat{\kappa}_+ &= \kappa_+(\hat{\rho}_+, \hat{\theta}_+), \quad \hat{\kappa}_- = \kappa_-(\hat{\theta}_-), \quad \hat{d}_+ = d_+(\hat{\rho}_+, \hat{\theta}_+), \quad \hat{d}_- = d_-(\hat{\theta}_-), \\ \tilde{\rho}_+ &= \hat{\rho}_+ - \rho_{*+}, \quad \tilde{\mu}_\pm = \hat{\mu}_\pm - \mu_{*\pm}, \quad \tilde{\lambda}_+ = \hat{\lambda}_+ - \lambda_{*+}, \quad \tilde{\kappa}_\pm = \hat{\kappa}_\pm - \kappa_{*\pm}, \\ \tilde{d}_\pm &= \hat{d}_\pm - d_{*\pm}.\end{aligned}$$

Setting $H_0 = \partial_t H$, $H_j = \partial_j H$ ($j = 1, \dots, N$), we have

$$\begin{aligned} (\partial_t f)(x', x_N + H(x, t), t) &= \partial_t \hat{f}(x, t) - \frac{H_0}{1 + H_N} \partial_N \hat{f}(x, t), \\ (\partial_j f)(x', x_N + H(x, t), t) &= \partial_j \hat{f}(x, t) - \frac{H_j}{1 + H_N} \partial_N \hat{f}(x, t) \quad (j = 1, \dots, N). \end{aligned} \quad (3.8)$$

In the following, we set

$$K_j = \frac{H_j}{1 + H_N} \quad (j = 0, 1, \dots, N), \quad \mathbf{K} = (K_1, \dots, K_N), \quad \mathbf{K}_0 = (K_0, \mathbf{K}).$$

By (3.8) we have

$$\nabla \pi_- = \mathbf{Q} \nabla \hat{\pi}_- = \begin{pmatrix} 1 & 0 & \cdots & 0 & K_1 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & K_{N-1} \\ 0 & 0 & \cdots & 0 & \frac{1}{1+H_N} \end{pmatrix} \begin{pmatrix} \partial_1 \hat{\pi}_- \\ \vdots \\ \partial_N \hat{\pi}_- \end{pmatrix},$$

and \mathbf{Q}^{-1} is given by

$$\mathbf{Q}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & -H_1 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -H_{N-1} \\ 0 & 0 & \cdots & 0 & 1 + H_N \end{pmatrix} = \mathbf{I} + \mathbf{Q}_1 \quad \text{with} \quad \mathbf{Q}_1 = \begin{pmatrix} 0 & \cdots & 0 & -H_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -H_{N-1} \\ 0 & \cdots & 0 & H_N \end{pmatrix}.$$

By (3.8) we have

$$\begin{aligned} \operatorname{div} \mathbf{u}_\pm &= \operatorname{div} \hat{\mathbf{u}}_\pm + V_{\operatorname{div}}(\hat{\mathbf{u}}_\pm, H) \\ &= \frac{1}{1 + H_N} \{ \operatorname{div} \hat{\mathbf{u}}_\pm - f_-(\hat{\mathbf{u}}_\pm, H) \} \\ &= \frac{1}{1 + H_N} \{ \operatorname{div} \hat{\mathbf{u}}_\pm - \operatorname{div} \mathbf{f}_-(\hat{\mathbf{u}}_\pm, H) \} \end{aligned} \quad (3.9)$$

with

$$\begin{aligned} V_{\operatorname{div}}(\hat{\mathbf{u}}_\pm, H) &= - \sum_{j=1}^N K_j \partial_N \hat{u}_{\pm j}, \quad f_-(\hat{\mathbf{u}}_\pm, H) = \sum_{j=1}^{N-1} (H_N \partial_j \hat{u}_{\pm j} - H_j \partial_N \hat{u}_{\pm j}), \\ \mathbf{f}_-(\hat{\mathbf{u}}_\pm, H) &= -(H_N \hat{u}_{\pm 1}, \dots, H_N \hat{u}_{\pm N-1}, - \sum_{j=1}^{N-1} H_j \hat{u}_{\pm j}). \end{aligned}$$

For any $N \times N$ matrix of functions $\mathbf{G} = {}^T(\mathbf{g}_1, \dots, \mathbf{g}_N)$, where ${}^T \mathbf{M}$ denotes the transposed \mathbf{M} , with ${}^T \mathbf{g}_i = (g_{i1}, \dots, g_{iN})$, by (3.9) we have

$$\operatorname{Div} \mathbf{G} = \operatorname{Div} \hat{\mathbf{G}} + \mathbf{V}_{\operatorname{Div}}(\hat{\mathbf{G}}, H) \quad (3.10)$$

with $\mathbf{V}_{\operatorname{Div}}(\hat{\mathbf{G}}, H) = {}^T(V_{\operatorname{div}}(\hat{\mathbf{g}}_1, H), \dots, V_{\operatorname{div}}(\hat{\mathbf{g}}_N, H))$. Moreover, we set

$$D_{ij}(\mathbf{u}_\pm) = D_{ij}(\hat{\mathbf{u}}_\pm) + V_{D_{ij}}(\hat{\mathbf{u}}_\pm, H), \quad \mathbf{D}(\mathbf{u}_\pm) = \mathbf{D}(\hat{\mathbf{u}}_\pm) + \mathbf{V}_{\mathbf{D}}(\hat{\mathbf{u}}_\pm, H), \quad (3.11)$$

where $V_{D_{ij}}(\hat{\mathbf{u}}_{\pm}, H) = -(K_i \partial_N \hat{u}_{\pm j} + K_j \partial_N \hat{u}_{\pm i})$ and $\mathbf{V}_D(\hat{\mathbf{u}}_{\pm}, H)$ is the $N \times N$ matrix whose (i, j) component is $V_{D_{ij}}(\hat{\mathbf{u}}_{\pm}, H)$.

Under these preparations, we see easily that problem (3.2), (3.3) and (3.4) is transformed to the following problem:

$$\begin{cases}
\begin{cases}
\partial_t \hat{\rho}_+ + \mathbf{v}_+ \cdot \nabla \hat{\rho}_+ + \hat{\rho}_+ (\operatorname{div} \hat{\mathbf{u}}_+ + V_{\operatorname{div}}(\hat{\mathbf{u}}_+, H)) = 0 \\
\rho_{*+} \partial_t \hat{\mathbf{u}}_+ - \operatorname{Div} \mathbf{S}_{*+}(\hat{\mathbf{u}}_+) = \mathbf{F}_+ \\
\rho_{*+} \kappa_{*+} \partial_t \hat{\theta}_+ - d_{*+} \Delta \hat{\theta}_+ = F_{\theta+}
\end{cases} & \text{in } \mathbb{R}_+^N \times (0, T), \\
\begin{cases}
\rho_{*-} \partial_t \hat{\mathbf{u}}_- - \operatorname{Div} \mathbf{S}_{*-}(\hat{\mathbf{u}}_-) + \nabla \hat{\pi}_- = \mathbf{F}_- \\
\operatorname{div} \hat{\mathbf{u}}_- = f_- = \operatorname{div} \mathbf{f}_- \\
\rho_{*-} \kappa_{*-} \partial_t \hat{\theta}_- - d_{*-} \Delta \hat{\theta}_- = F_{\theta-}
\end{cases} & \text{in } \mathbb{R}_-^N \times (0, T), \\
\begin{cases}
\mu_{*-} D_{iN}(\hat{\mathbf{u}}_-)|_- - \mu_{*+} D_{iN}(\hat{\mathbf{u}}_+)|_+ = G_i \\
T_- - T_+ = \sigma \Delta' H + G_N, \quad \rho_{*-}^{-1} T_- - \rho_{*+}^{-1} T_+ = G_{N+1} \\
\hat{u}_{-i}|_- - \hat{u}_{+i}|_+ = K_i \\
\hat{\theta}_-|_- - \hat{\theta}_+|_+ = 0, \quad d_{*-} \partial_N \hat{\theta}_-|_- - d_{*+} \partial_N \hat{\theta}_+|_+ = G_{\theta} \\
\partial_t H - \left(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} \hat{u}_{-N}|_- - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} \hat{u}_{+N}|_+ \right) = G_h
\end{cases} & \text{on } \mathbb{R}_0^N \times (0, T), \\
\begin{cases}
(\hat{\rho}_+, \hat{\mathbf{u}}_+, \hat{\theta}_+)|_{t=0} = (\hat{\rho}_{0+}, \hat{\mathbf{u}}_{0+}, \hat{\theta}_{0+}) & \text{in } \mathbb{R}_+^N, \\
(\hat{\mathbf{u}}_-, \hat{\theta}_-)|_{t=0} = (\hat{\mathbf{u}}_{0-}, \hat{\theta}_{0-}) & \text{in } \mathbb{R}_-^N, \quad H|_{t=0} = H_0 & \text{on } \mathbb{R}_0^N
\end{cases}
\end{cases} \quad (3.12)$$

where $i = 1, \dots, N-1$, and we have set

$$\begin{aligned}
T_- &= (\mu_{*-} D_{NN}(\hat{\mathbf{u}}_-) - \hat{\pi}_-)|_-, \\
T_+ &= (\mu_{*+} D_{NN}(\hat{\mathbf{u}}_+) + (\lambda_{*+} - \mu_{*+}) \operatorname{div} \hat{\mathbf{u}}_+), \\
\mathbf{v}_+ &= (\hat{u}_{+1}, \dots, \hat{u}_{+N-1}, \hat{u}_{+N} - K_0 - \sum_{j=1}^N K_j \hat{u}_{+j}), \\
\mathbf{S}_{*+}(\mathbf{u}) &= \mu_{*+} \mathbf{D}(\mathbf{u}) + (\lambda_{*+} - \mu_{*+}) \operatorname{div} \mathbf{u} \mathbf{I}, \quad \mathbf{S}_{*-}(\mathbf{u}) = \mu_{*-} \mathbf{D}(\mathbf{u}), \\
\hat{\rho}_{0+}(x) &= \rho_{0+}(x', x_N + H_0(x)), \quad \hat{\mathbf{u}}_{0\pm}(x) = \mathbf{u}_{0\pm}(x', x_N + H_0(x)), \\
\hat{\theta}_{0\pm}(x) &= \theta_{0\pm}(x', x_N + H_0(x)).
\end{aligned} \quad (3.14)$$

Here, $f|_{\pm}(x_0) = \lim_{\substack{x \rightarrow x_0 \\ x \in \Omega_{\pm}^{(t)}}} f(x)$ for $x_0 \in \mathbb{R}_0^N$. Moreover, H_0 is a function satisfying the equations $(1 - \Delta)H_0 = 0$ in \mathbb{R}_{\pm}^N and $H_0|_{x_N=0} = h_0$, and the right-hand sides in (3.12) are defined by the following formulas:

$$\begin{aligned}
\mathbf{F}_+ &= \mathbf{F}_+(\hat{\rho}_+, \hat{\mathbf{u}}_+, H) = \\
&= -\hat{\rho}_+ \partial_t \hat{\mathbf{u}}_+ + \hat{\rho}_+ \{K_0 \partial_N \hat{\mathbf{u}}_+ - \hat{\mathbf{u}}_+ \cdot \nabla \hat{\mathbf{u}}_+ + (\hat{\mathbf{u}}_+ \cdot \mathbf{K}) \partial_N \hat{\mathbf{u}}_+\} \\
&+ \operatorname{Div}(\tilde{\mu}_+ \mathbf{D}(\hat{\mathbf{u}}_+) + \hat{\mu}_+ \mathbf{V}_D(\hat{\mathbf{u}}_+, H)) + \mathbf{V}_D(\hat{\mu}_+ (\mathbf{D}(\hat{\mathbf{u}}_+) + \mathbf{V}_D(\hat{\mathbf{u}}_+, H))) \\
&+ \nabla \{(\tilde{\lambda}_+ - \tilde{\mu}_+) \operatorname{div} \hat{\mathbf{u}}_+ + (\hat{\lambda}_+ - \hat{\mu}_+) V_{\operatorname{div}}(\hat{\mathbf{u}}_+, H)\} \\
&+ \mathbf{V}_{\operatorname{Div}}((\hat{\lambda}_+ - \hat{\mu}_+) (\operatorname{div} \hat{\mathbf{u}}_+ + V_{\operatorname{div}}(\hat{\mathbf{u}}_+, H)) \mathbf{I}) - \mathbf{Q} \nabla P_+(\hat{\rho}_+, \hat{\theta}_+), \\
F_{\theta+} &= F_{\theta+}(\hat{\rho}_+, \hat{\mathbf{u}}_+, \hat{\theta}_+, H) = -d_{*+} \sum_{j=1}^N \partial_j (K_j \partial_N \hat{\theta}_+)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^N \partial_j (\tilde{d}_+ (\partial_j \hat{\theta}_+ - K_j \partial_N \hat{\theta}_+)) + \sum_{j=1}^N K_j \partial_N (\hat{d}_+ (\partial_j \hat{\theta}_+ - K_j \partial_N \hat{\theta}_+)) \\
& - (\hat{\rho}_+ \hat{\kappa}_+ - \rho_{*+} \kappa_{*+}) \partial_t \hat{\theta}_+ + \hat{\rho}_+ \hat{\kappa}_+ (K_0 \partial_N \hat{\theta}_+ - \hat{\mathbf{u}}_+ \cdot \nabla \hat{\theta}_+ + (\hat{\mathbf{u}}_+ \cdot \mathbf{K}) \partial_N \hat{\theta}_+) \\
& + 2\hat{\mu}_+ |\mathbf{D}(\hat{\mathbf{u}}_+) + \mathbf{V}_\mathbf{D}(\hat{\mathbf{u}}_+, H)|^2 + (\hat{\lambda}_+ - \hat{\mu}_+) (\operatorname{div} \hat{\mathbf{u}}_+ + V_{\operatorname{div}}(\hat{\mathbf{u}}_+, H))^2 \\
& + P_+(\hat{\rho}_+, \hat{\theta}_+) (1 - \frac{1}{\hat{\rho}_+}) (\operatorname{div} \hat{\mathbf{u}}_+ + V_{\operatorname{div}}(\hat{\mathbf{u}}_+, H)), \\
\mathbf{F}_- &= \mathbf{F}_-(\hat{\mathbf{u}}_-, H) = -\mathbf{Q}_1(\rho_{*-} \partial_t \hat{\mathbf{u}}_- - \mu_{*-} \operatorname{Div} \mathbf{D}(\hat{\mathbf{u}}_-)) \\
& - (I + \mathbf{Q}_1) \{ \rho_{*-} (K_0 \partial_N \hat{\mathbf{u}}_- - \hat{\mathbf{u}}_- \cdot \nabla \hat{\mathbf{u}}_- + (\hat{\mathbf{u}}_- \cdot \mathbf{K}) \partial_N \hat{\mathbf{u}}_-) \\
& + \operatorname{Div}(\tilde{\mu}_- \mathbf{D}(\hat{\mathbf{u}}_-) + \hat{\mu}_- \mathbf{V}_\mathbf{D}(\hat{\mathbf{u}}_-, H)) + \mathbf{V}_{\operatorname{Div}}(\hat{\mu}_- (\operatorname{Div} \mathbf{D}(\hat{\mathbf{u}}_-) + \mathbf{V}_\mathbf{D}(\hat{\mathbf{u}}_-, H))) \}, \\
f_- &= f_-(\hat{\mathbf{u}}_-, H) = \sum_{j=1}^{N-1} \{ H_N \partial_j \hat{u}_{-j} - H_j \partial_N \hat{u}_{-j} \}, \\
\mathbf{f}_- &= \mathbf{f}_-(\hat{\mathbf{u}}_-, H) = -(H_N \hat{u}_{-1}, \dots, H_N \hat{u}_{-N-1}, - \sum_{j=1}^{N-1} H_j \hat{u}_{-j}), \\
F_{\theta_-} &= F_{\theta_-}(\hat{\mathbf{u}}_-, \hat{\theta}_-, H) = -\rho_{*-} \tilde{\kappa}_- \partial_t \hat{\theta}_- - d_{*-} \sum_{j=1}^N \partial_j (K_j \partial_N \hat{\theta}_-) \\
& + \rho_{*-} \hat{\kappa}_- (K_0 \partial_N \hat{\theta}_- - \hat{\mathbf{u}}_- \cdot \nabla \hat{\theta}_- + (\hat{\mathbf{u}}_- \cdot \mathbf{K}) \partial_N \hat{\theta}_-) + \sum_{j=1}^N \partial_j (\tilde{d}_+ (\partial_j \hat{\theta}_- - K_j \partial_N \hat{\theta}_-)) \\
& + \sum_{j=1}^N K_j \partial_N (\hat{d}_- (\partial_j \hat{\theta}_- - K_j \partial_N \hat{\theta}_-)) + 2\hat{\mu}_- |\mathbf{D}(\hat{\mathbf{u}}_-) + \mathbf{V}_\mathbf{D}(\hat{\mathbf{u}}_-, H)|^2, \\
G_i &= G_i(\hat{\rho}_+, \hat{\mathbf{u}}_{\pm}, H) = \\
& - \{ (\tilde{\mu}_- D_{iN}(\hat{\mathbf{u}}_-) + \hat{\mu}_- V_{D_{iN}}(\hat{\mathbf{u}}_-, H))|_- - (\tilde{\mu}_+ D_{iN}(\hat{\mathbf{u}}_+) + \hat{\mu}_+ V_{D_{iN}}(\hat{\mathbf{u}}_+, H))|_+ \} \\
& + \sum_{j=1}^{N-1} (\partial_j H) \{ \hat{\mu}_- (D_{ij}(\hat{\mathbf{u}}_-) + V_{D_{ij}}(\hat{\mathbf{u}}_-, H))|_- - \hat{\mu}_+ (D_{ij}(\hat{\mathbf{u}}_+) + V_{D_{ij}}(\hat{\mathbf{u}}_+, H))|_+ \} \\
& - (\partial_i H) [\sum_{j=1}^{N-1} (\partial_j H) \{ \hat{\mu}_- (D_{ij}(\hat{\mathbf{u}}_-) + V_{D_{ij}}(\hat{\mathbf{u}}_-, H))|_- - \hat{\mu}_+ (D_{ij}(\hat{\mathbf{u}}_+) + V_{D_{ij}}(\hat{\mathbf{u}}_+, H))|_+ \}, \\
& - \{ \hat{\mu}_- (D_{NN}(\hat{\mathbf{u}}_-) + V_{D_{NN}}(\hat{\mathbf{u}}_-, H))|_- - \hat{\mu}_+ (D_{NN}(\hat{\mathbf{u}}_+) + V_{D_{NN}}(\hat{\mathbf{u}}_+, H))|_+ \}, \\
G_N &= G_N(\hat{\rho}_+, \hat{\mathbf{u}}_{\pm}, H) = \\
& - (\tilde{\mu}_- D_{NN}(\hat{\mathbf{u}}_-) + \hat{\mu}_- V_{D_{NN}}(\hat{\mathbf{u}}_-, H))|_- - (\tilde{\mu}_+ D_{NN}(\hat{\mathbf{u}}_+) + \hat{\mu}_+ V_{D_{NN}}(\hat{\mathbf{u}}_+, H))|_+ \\
& + \{ (\tilde{\lambda}_+ - \tilde{\mu}_+) \operatorname{div} \mathbf{u}_+ + (\hat{\lambda}_+ - \hat{\mu}_+) V_{\operatorname{div}}(\hat{\mathbf{u}}_+, H) - (P_+(\hat{\rho}_+, \hat{\theta}_+) - P_+(\rho_{*+}, \theta_{*+})) \}|_+ \\
& + \sum_{j=1}^{N-1} (\partial_j H) (\hat{\mu}_- (D_{Nj}(\hat{\mathbf{u}}_-) + V_{D_{Nj}}(\hat{\mathbf{u}}_-, H))|_- - \hat{\mu}_+ (D_{Nj}(\hat{\mathbf{u}}_+) + V_{D_{Nj}}(\hat{\mathbf{u}}_+, H))|_+) \\
& + \sigma \left\{ \left(1 - \frac{1}{\sqrt{1 + |\nabla' H|^2}} \Delta' H - \sum_{i,j=1}^{N-1} \frac{(\partial_i H)(\partial_i \partial_j H)}{(1 + |\nabla' H|^2)^{3/2}} \right) \right. \\
& \left. + \left(\frac{1}{\rho_{*-}} - \frac{1}{\hat{\rho}_+|_+} \right)^{-1} (\hat{u}_{-N}|_- - \hat{u}_{+N}|_+)^2 (1 + |\nabla' H|^2), \right.
\end{aligned}$$

$$\begin{aligned}
G_{N+1} &= G_{N+1}(\hat{\rho}_+, \hat{\mathbf{u}}_{\pm}, \hat{\theta}_{\pm}, H) = \\
& - \frac{1}{\rho_{*-}}(P_+(\hat{\rho}_+, \hat{\theta}_+) - P_+(\rho_{*+}, \theta_{*+})) - \left(\frac{1}{\hat{\rho}_+|_+} - \frac{1}{\rho_{*+}} \right) P_+(\hat{\rho}_+, \hat{\theta}_+) \\
& + (\psi_-(\hat{\theta}_-) - \psi_-(\theta_*))|_- - (\psi_+(\hat{\rho}_+, \hat{\theta}_+) - \psi_+(\rho_{*+}, \theta_*))|_+ \\
& + (\psi_-(\hat{\theta}_-)|_- - \psi_+(\hat{\rho}_+, \hat{\theta}_+)|_+)|\nabla' H|^2 \\
& + \frac{1}{2} \left(\frac{1}{\rho_{*-}} + \frac{1}{\hat{\rho}_+|_+} \right) \left(\frac{1}{\rho_{*-}} - \frac{1}{\hat{\rho}_+|_+} \right)^{-1} (u_{-N}|_- - u_{+N}|_+)^2 (1 + |\nabla' H|^2)^2 \\
& - \left\{ \frac{1}{\rho_{*-}} \tilde{\mu}_- D_{NN}(\hat{\mathbf{u}}_-)|_- - \frac{1}{\rho_{*+}} (\tilde{\mu}_+ D_{NN}(\hat{\mathbf{u}}_+) + (\tilde{\lambda}_+ - \tilde{\mu}_+) \operatorname{div} \mathbf{u}_+)|_+ \right\} \\
& + \left(\frac{1}{\hat{\rho}_+|_+} - \frac{1}{\rho_{*+}} \right) \{ \hat{\mu}_+ (D_{NN}(\hat{\mathbf{u}}_+) + V_{D_{NN}}(\hat{\mathbf{u}}_+, H)) \\
& \quad + (\hat{\lambda}_+ - \hat{\mu}_+) (\operatorname{div} \hat{\mathbf{u}}_+ + V_{\operatorname{div}}(\hat{\mathbf{u}}_+, H)) \} |_+ \\
& - \sum_{i,j=1}^{N-1} \left\{ \frac{\hat{\mu}_-}{\rho_{*-}} (D_{ij}(\hat{\mathbf{u}}_-) + V_{D_{ij}}(\hat{\mathbf{u}}_-, H))|_- \right. \\
& \quad \left. - \frac{\hat{\mu}_+}{\hat{\rho}_+} (D_{ij}(\hat{\mathbf{u}}_+) + V_{D_{ij}}(\hat{\mathbf{u}}_+, H))|_+ \right\} (\partial_i H)(\partial_j H) \\
& + 2 \sum_{i=1}^{N-1} \left\{ \frac{\hat{\mu}_-}{\rho_{*-}} (D_{iN}(\hat{\mathbf{u}}_-) + V_{D_{iN}}(\hat{\mathbf{u}}_-, H))|_- \right. \\
& \quad \left. - \frac{\hat{\mu}_+}{\hat{\rho}_+} (D_{iN}(\hat{\mathbf{u}}_+) + V_{D_{iN}}(\hat{\mathbf{u}}_+, H))|_+ \right\} (\partial_i H) \\
& - |\nabla' H|^2 \left[\sigma \left(\frac{\Delta' H}{\sqrt{1 + |\nabla' H|^2}} - \sum_{i,j=1}^{N-1} \frac{(\partial_i H)(\partial_i \partial_j H)}{(1 + |\nabla' H|^2)^{3/2}} \right) \right. \\
& \quad \left. + \left(\frac{1}{\rho_{*-}} - \frac{1}{\hat{\rho}_+|_+} \right)^{-1} (u_{-N}|_- - u_{+N}|_+)^2 (1 + |\nabla' H|^2) \right. \\
& \quad \left. + \sum_{i=1}^{N-1} \{ \hat{\mu}_- (D_{iN}(\hat{\mathbf{u}}_-) + V_{D_{iN}}(\hat{\mathbf{u}}_-, H))|_- - \hat{\mu}_+ (D_{iN}(\hat{\mathbf{u}}_+) + V_{D_{iN}}(\hat{\mathbf{u}}_+, H))|_+ \} (\partial_i H) \right. \\
& \quad \left. - \{ \hat{\mu}_- D_{NN}(\hat{\mathbf{u}}_-) + V_{D_{NN}}(\hat{\mathbf{u}}_-, H)|_- - \hat{\mu}_+ (D_{NN}(\hat{\mathbf{u}}_+) + V_{D_{NN}}(\hat{\mathbf{u}}_+, H))|_+ \} \right], \\
K_i &= K_i(\hat{\mathbf{u}}_{\pm}, H) = -(\partial_i H)(\hat{u}_{-N}|_- - \hat{u}_{+N}|_+), \\
G_{\theta} &= G_{\theta}(\hat{\rho}_+, \hat{\mathbf{u}}_{\pm}, \hat{\theta}_{\pm}, H) = \\
& (1 + |\nabla' H|^2)(\hat{u}_{-N}|_- - \hat{u}_{+N}|_+) \left(\frac{1}{\rho_{*-}} - \frac{1}{\hat{\rho}_+|_+} \right)^{-1} (\hat{\theta}_- \eta_- (\hat{\theta}_-)|_- - \hat{\theta}_+ \eta_+ (\hat{\rho}_+, \hat{\theta}_+)|_+) \\
& - (\tilde{d}_- (\nabla \hat{\theta}_- - \mathbf{K} \partial_N \hat{\theta}_-)|_- - \tilde{d}_+ (\nabla \hat{\theta}_+ - \mathbf{K} \partial_N \hat{\theta}_+)|_+) \cdot (-\nabla' H, 1) \\
& + (d_{*-} \nabla' \hat{\theta}_-|_- - d_{*+} \nabla' \hat{\theta}_+|_+) \cdot \nabla' H + (d_{*-} \partial_N \hat{\theta}_-|_- - d_{*+} \partial_N \hat{\theta}_+|_+) \mathbf{K} \cdot (-\nabla' H, 1), \\
G_h &= G_h(\hat{\rho}_+, \hat{\mathbf{u}}_{\pm}, H) = \\
& \left(\frac{1}{\rho_{*-} - \hat{\rho}_+|_+} - \frac{1}{\rho_{*-} - \rho_{*+}} \right) (\rho_{*-} \hat{u}_{-N}|_- - \rho_{*+} \hat{u}_{+N}|_+) + \frac{\rho_{*+} - \hat{\rho}_+}{\rho_{*-} - \hat{\rho}_+} \hat{u}_{+N}|_+ \\
& - \frac{\rho_{*-}}{\rho_{*-} - \hat{\rho}_+|_+} \sum_{j=1}^{N-1} (\partial_j H) \hat{u}_{-j}|_- + \frac{\hat{\rho}_+}{\rho_{*-} - \hat{\rho}_+} \sum_{j=1}^{N-1} (\partial_j H) \hat{u}_{+j}|_+.
\end{aligned}$$

The phase flux \mathbf{j} is eliminated by using the formula:

$$\mathbf{j} = (\hat{u}_{-N}|_- - \hat{u}_{+N}|_+) \left(\frac{1}{\rho_{*-}} - \frac{1}{\hat{\rho}_{+|_+} + \rho_{*+}} \right)^{-1} \sqrt{1 + |\nabla' H|^2} \quad \text{on } \mathbb{R}_0^N \times (0, T).$$

Moreover, we have used the formulas: for $x \in \Gamma(t)$ and $t > 0$

$$\begin{aligned} & \pi_- - P_+ + (\lambda_+ - \mu_+) \operatorname{div} \mathbf{u}_+ \\ &= -\sigma H_\Gamma - \left[\left[\frac{1}{\rho} \right] \right]^2 - \sum_{j=1}^{N-1} [[\mu D_{Nj}]] (\partial_j H) + [[\mu D_{NN}]], \\ & H_\Gamma \mathbf{n}_{\Gamma(t)} = \left\{ \operatorname{div}' \left(\frac{\nabla' H}{1 + |\nabla' H|^2} \right) \right\} (-\nabla' H, 1) / \sqrt{1 + |\nabla' H|^2} \end{aligned}$$

where $\nabla' H = (\partial_1 H, \dots, \partial_{N-1} H)$ and $\operatorname{div}' \mathbf{v}' = \sum_{j=1}^{N-1} \partial_j v_j$ for $\mathbf{v}' = (v_1, \dots, v_{N-1})$.

Since $T_- - T_+ = \sigma \Delta' H + G_N$ and $\rho_{*-}^{-1} T_- - \rho_{*+}^{-1} T_+ = G_{N+1}$ are equivalent to

$$T_\pm = \frac{\rho_\pm \sigma}{\rho_{*-} - \rho_{*+}} \Delta' H + \frac{\rho_{*-} \rho_{*+}}{\rho_{*-} - \rho_{*+}} (\rho_{*+}^{-1} G_N - G_{N+1}), \quad (3.15)$$

the compatibility condition for problem (3.12) is

$$\begin{aligned} & \operatorname{div} \hat{\mathbf{u}}_{-0} = f_- (\hat{\mathbf{u}}_{-0}, H_0) = \operatorname{div} \mathbf{f}_- (\hat{\mathbf{u}}_{-0}, H_0) \quad \text{in } \mathbb{R}_-^N, \\ & \mu_{*-} D_{iN} (\hat{\mathbf{u}}_{-0})|_- - \mu_{*+} D_{iN} (\hat{\mathbf{u}}_{+0})|_+ = G_i (\hat{\rho}_{0+}, \hat{\mathbf{u}}_{0\pm}, H_0) \quad (i = 1, \dots, N-1), \\ & \hat{u}_{0-i}|_- - \hat{u}_{0+i}|_+ = K_i (\hat{\mathbf{u}}_\pm, H_0) \quad (i = 1, \dots, N-1), \\ & \hat{\theta}_{0-}|_- - \hat{\theta}_{0+}|_+ = 0, \\ & d_{*-} \partial_N \hat{\theta}_{0-}|_- - d_{*+} \partial_N \hat{\theta}_{0+}|_+ = G_\theta (\hat{\rho}_{0+}, \hat{\mathbf{u}}_{0\pm}, \hat{\theta}_{0\pm}, H_0), \\ & (\mu_{*+} D_{NN} (\hat{\mathbf{u}}_{0+}) + (\lambda_{*+} - \mu_{*+}) \operatorname{div} \hat{\mathbf{u}}_{0+})|_+ = \frac{\rho_{*+} \sigma}{\rho_{*-} - \rho_{*+}} \Delta' h_0 \\ & + \frac{\rho_{*-} \rho_{*+}}{\rho_{*-} - \rho_{*+}} (\rho_{*-}^{-1} G_N (\hat{\rho}_+, \hat{\mathbf{u}}_{0\pm}, H_0) - G_{N+1} (\hat{\rho}_+, \hat{\mathbf{u}}_{0\pm}, \hat{\theta}_{0\pm}, H_0)). \end{aligned} \quad (3.16)$$

The following theorem is the main result of this paper concerning the local well-posedness of problem (3.12).

Theorem 3.1. *Let $1 < p, q < \infty$ with $2/p + N/q < 1$. Assume that $\rho_{*\pm}$ and θ_* satisfy the condition (3.5). Then, given any positive time T , there exists an $\epsilon > 0$ such that problem (3.12) admits unique solutions $\hat{\rho}_+$, $\hat{\mathbf{u}}_\pm$ and $\hat{\theta}_\pm$ with*

$$\begin{aligned} & \hat{\rho}_+ \in W_p^1((0, T), L_q(\mathbb{R}_+^N)) \cap L_p((0, T), W_q^1(\mathbb{R}_+^N)), \\ & (\hat{\mathbf{u}}_\pm, \hat{\theta}_\pm) \in W_p^1((0, T), L_q(\mathbb{R}_\pm^N)) \cap L_p((0, T), W_q^2(\mathbb{R}_\pm^N)), \\ & H \in W_p^1((0, T), W_q^2(\mathbb{R}^N)) \cap L_p((0, T), W_q^3(\mathbb{R}^N)) \end{aligned}$$

for any initial data

$$\hat{\rho}_{0+} \in W_q^1(\mathbb{R}_+^N), \quad (\hat{\mathbf{u}}_{0\pm}, \hat{\theta}_{0\pm}) \in B_{q,p}^{2(1-1/p)}(\mathbb{R}_\pm^N), \quad H_0 \in W_{q,p}^{3-1/p}(\mathbb{R}^N)$$

satisfying the smallness condition:

$$\|\hat{\rho}_{0+}\|_{W_q^1(\mathbb{R}_+^N)} + \sum_{\ell=\pm} \|(\hat{\mathbf{u}}_{0\ell}, \hat{\theta}_{0\ell})\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_\ell^N)} + \|H_0\|_{W_{q,p}^{3-1/p}(\mathbb{R}^N)} \leq \epsilon$$

and compatibility condition (3.16).

Here and in the following, $L_q(\mathbb{R}_\pm^N)$ and $W_q^m(\mathbb{R}_\pm^N)$ denote the usual Lebesgue space and Sobolev space of order m in the $L_q(\mathbb{R}_\pm^N)$ sense, while $\|\cdot\|_{L_q(\mathbb{R}_\pm^N)}$ and $\|\cdot\|_{W_q^m(\mathbb{R}_\pm^N)}$ denote their norms, respectively. For the Banach space X , $L_p((0, T), X)$ and $W_p^m((0, T), X)$ denote the Lebesgue space and the Sobolev space with values in X , while $\|\cdot\|_{L_p((0, T), X)}$ and $\|\cdot\|_{W_p^m((0, T), X)}$ denote their norms, respectively. $B_{q,p}^{(1-\theta)a+\theta b}(\mathbb{R}_\pm^N)$ denotes the real interpolation space $(W_q^a(\mathbb{R}_\pm^N), W_q^b(\mathbb{R}_\pm^N))_{\theta,p}$ with real interpolation functor $(\cdot, \cdot)_{\theta,p}$ and $0 < \theta < 1$.

Remark 2. (1) The mathematical study of the compressible and incompressible two phase problem is quite rare as far as the author knows. First Denisova [2] studied the evolution of the compressible and incompressible two phase flow with sharp interface without phase transition under some restriction on the viscosity coefficients. Recently, Kubo, Shibata and Soga [4] studied the same problem as in [2] without any restriction on viscosity coefficients in case of without surface tension and without phase transition. This paper is the first manuscript to treat the compressible and incompressible two phase problem with phase transition *. The incompressible and incompressible two phase problem with phase transition was studied by J. Prüss, et al. [6, 7, 8].

4 Maximal L_p - L_q regularity

In the following, we assume that $N < q < \infty$ in view of the Sobolev imbedding theorem: $\|v\|_{L_\infty(\Omega)} \leq C\|v\|_{W_q^1(\Omega)}$ with $\Omega = \mathbb{R}_\pm^N$ and $\Omega = \mathbb{R}^N$. To solve problem (3.12), we use the maximal L_p - L_q regularity for the parabolic equations. From this point of view, we represent $\hat{\rho}_+$ by the integration along the characteristic curve generated by \mathbf{v}_+^\dagger to eliminate $\hat{\rho}_+$ from the first equation of (3.12), which is the hyperbolic equation for $\hat{\rho}_+$.

Given function f defined on \mathbb{R}_+^N , the Lions extension $\text{Ext}[f]$ of f is defined by

$$\text{Ext}[f](x, t) = \begin{cases} f(x, t) & \text{for } x_N > 0, \\ 3f(x', -x_N, t) - 2f(x', -2x_N, t) & \text{for } x_N < 0, \end{cases}$$

Set $\hat{\mathbf{w}}_+ = \text{Ext}[\hat{\mathbf{u}}_+]$ and $\mathbf{v} = (\hat{w}_{+1}, \dots, \hat{w}_{+N-1}, \hat{w}_{+N} - K_0 - \sum_{j=1}^N K_j \hat{w}_{+j})$. Note that $\mathbf{v} = \mathbf{v}_+$ on \mathbb{R}_+^N . We assume that

$$\int_0^T \|\nabla \mathbf{v}(\cdot, t)\|_{L_\infty(\mathbb{R}^N)} dt \leq \epsilon_1 \quad (4.1)$$

with some small positive constant $\epsilon_1 > 0$. We use the usual fixed point argument to solve the nonlinear problem and in this argument we keep the situation where $\hat{\mathbf{u}}_+$ and H satisfy (4.1).

*Modeling and the main results of this paper were announced in the abstract of 39th Sapporo symposium on PDE at Hokkaido University (cf. [13]).

[†]Tani [17] represented the mass density with the help of the velocity field to prove the local well-posedness of the Navier-Stokes equations describing the compressible viscous fluid flow (cf. also [16, 18]) It was also suggested by J. Prüss to the author to represent $\hat{\rho}_+$ by $\hat{\mathbf{u}}_+$ and H with the help of the equation of balance of mass when the author visited Halle university in the early of April, 2014.

Let $\hat{\xi}$ be the solution to the Cauchy problem:

$$\frac{d}{dt}\hat{\xi}(\eta, t) = \mathbf{v}(\hat{\xi}(\eta, t), t), \quad \hat{\xi}(\eta, 0) = \eta \in \mathbb{R}^N.$$

According to Ströhmer [15], we choose an $\epsilon_1 > 0$ so small that the map: $\eta \mapsto \xi$ is bijective on \mathbb{R}^N for any $t \in [0, T]$. We denote its inverse map by $\hat{\eta} = \hat{\eta}(\xi, t)$. We look for $\hat{\rho}_+$ satisfying the equation:

$$\partial_t \hat{\rho}_+ + \mathbf{v} \cdot \nabla \hat{\rho}_+ + \hat{\rho}_+ (\operatorname{div} \hat{\mathbf{w}}_+ + V_{\operatorname{div}}(\hat{\mathbf{w}}_+, H)) = 0 \quad \text{in } \mathbb{R}^N \times (0, T). \quad (4.2)$$

Since

$$\frac{\partial}{\partial t} \hat{\rho}_+(\hat{\xi}(\eta, t), t) = (\partial_t \hat{\rho}_+ + \mathbf{v} \cdot \nabla \hat{\rho}_+)(\hat{\xi}(\eta, t), t) = g(\hat{\xi}(\eta, t), t) \rho_+(\hat{\xi}(\eta, t), t),$$

with $g = -(\operatorname{div} \hat{\mathbf{w}}_+ + V_{\operatorname{div}}(\hat{\mathbf{w}}_+, H))$, defining $\hat{\rho}_+(\xi, t)$ by

$$\hat{\rho}_+(\xi, t) = (\rho_{*+} + \tilde{\rho}_{0+}(\eta)) e^{-\int_0^t (\operatorname{div} \hat{\mathbf{w}}_+ + V_{\operatorname{div}}(\hat{\mathbf{w}}_+, H))(\hat{\xi}(\eta, s), s) ds} \quad (4.3)$$

with $\eta = \hat{\eta}(\xi, t)$, where $\tilde{\rho}_{0+}(\eta) = \operatorname{Ext}[\hat{\rho}_{0+}]$ to \mathbb{R}^N , we see that $\hat{\rho}_+$ is a required function satisfying (4.2) with $\hat{\rho}_+(\xi, 0) = \rho_{*+} + \hat{\rho}_{0+}(\xi)$ in \mathbb{R}_+^N .

Inserting the formula of $\hat{\rho}_+$ given in (4.3) into the right-hand sides: $\mathbf{F}_+ = \mathbf{F}_+(\hat{\rho}_+, \hat{\mathbf{u}}_{\pm}, H)$, $F_{\theta+} = F_{\theta+}(\hat{\rho}_+, \hat{\mathbf{u}}_{\pm}, \hat{\theta}_+, H)$, $G_j = G_j(\hat{\rho}_+, \mathbf{u}_{\pm}, H)$ ($j = 1, \dots, N+1$) and $G_{\theta} = G_{\theta}(\hat{\rho}_+, \hat{\mathbf{u}}_{\pm}, \hat{\theta}_{\pm}, H)$ in (3.12), we have the interface problem of the final form, which is a quasilinear parabolic equation. As the linearized problem, we have the decoupled two systems. One is the Stokes equation with interface condition:

$$\begin{cases} \rho_{*+} \partial_t \mathbf{u}_+ - \operatorname{Div} \mathbf{S}_{*+}(\mathbf{u}_+) = \mathbf{f}_+ & \text{in } \mathbb{R}_+^N \times (0, T) \\ \rho_{*-} \partial_t \mathbf{u}_- - \operatorname{Div} \mathbf{S}_{*-}(\mathbf{u}_-) + \nabla \pi_- = \mathbf{f}_- & \text{in } \mathbb{R}_-^N \times (0, T) \\ \operatorname{div} \mathbf{u}_- = f_{\operatorname{div}} = \operatorname{div} \mathbf{f}_{\operatorname{div}} & \end{cases} \quad (4.4)$$

subject to the interface condition: for $x \in \mathbb{R}_0^N$ and $t \in (0, T)$

$$\begin{aligned} \mu_{*-} D_{iN}(\mathbf{u}_-)|_- - \mu_{*+} D_{iN}(\mathbf{u}_+)|_+ &= g_i \quad (i = 1, \dots, N-1), \\ (\mu_{*-} D_{NN}(\mathbf{u}_-) - \pi_-)|_- &= \sigma_- \Delta' H + g_N \\ (\mu_{*+} D_{NN}(\mathbf{u}_+) + (\lambda_{*+} - \mu_{*+}) \operatorname{div} \mathbf{u}_+)|_+ &= \sigma_+ \Delta' H + g_{N+1}, \\ u_{-i}|_- - u_{+i}|_+ &= h_i \quad (i = 1, \dots, N-1), \\ \partial_t H - \left(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} u_{-N} - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} u_{+N} \right) &= d \end{aligned} \quad (4.5)$$

and the initial condition:

$$\mathbf{u}_{\pm}|_{t=0} = \mathbf{u}_{0\pm} \quad \text{in } \mathbb{R}_{\pm}^N, \quad H|_{t=0} = H_0 \quad \text{in } \mathbb{R}_0^N, \quad (4.6)$$

where we have set $\sigma_{\pm} = \rho_{\pm} \sigma(\rho_{*-} - \rho_{*+})^{-1}$ and we have used the equivalent relations (3.15). Another is the heat equations with interface condition:

$$\begin{aligned} \rho_{*+} \kappa_{*+} \partial_t \theta_+ - d_{*+} \Delta \theta_+ &= \tilde{f}_+ \quad \text{in } \mathbb{R}_+^N \times (0, T) \\ \rho_{*-} \kappa_{*-} \partial_t \theta_- - d_{*-} \Delta \theta_- &= \tilde{f}_- \quad \text{in } \mathbb{R}_-^N \times (0, T) \end{aligned} \quad (4.7)$$

subject to the interface condition: for $x \in \mathbb{R}_0^N$ and $t \in (0, T)$

$$\theta_-|_- - \theta_+|_+ = 0, \quad d_{*+}\partial_N\theta_-|_- - d_{*+}\partial_N\theta_+|_+ = \tilde{g} \quad (4.8)$$

and the initial condition:

$$\theta_\pm|_{t=0} = \theta_{0\pm} \quad \text{on } \mathbb{R}_\pm^N. \quad (4.9)$$

We have the following theorem about the maximal L_p - L_q regularity for problem (4.4), (4.5), (4.6).

Theorem 4.1. *Let $1 < p, q < \infty$ and $0 < T < \infty$. Assume that $\rho_{*-} \neq \rho_{*+}$. Then, for any initial data $\mathbf{u}_{0\pm} \in B_{q,p}^{2(1-1/p)}(\mathbb{R}_\pm^N)$ and $H_0 \in B_{q,p}^{3-1/p}(\mathbb{R}^N)$, and right-hand sides of (4.4) and (4.5)*

$$\begin{aligned} \mathbf{f}_\pm &\in L_p((0, T), L_q(\mathbb{R}_\pm^N)), f_{\text{div}} \in L_p((0, T), W_q^1(\mathbb{R}_-^N)), \mathbf{f}_{\text{div}} \in W_p^1((0, T), L_q(\mathbb{R}_-^N)) \\ d &\in L_p((0, T), W_q^2(\mathbb{R}^N)), g_i \in L_p((0, T), W_q^1(\mathbb{R}^N)) \cap W_p^1((0, T), W_q^{-1}(\mathbb{R}^N)) \\ h_j &\in L_p((0, T), W_q^2(\mathbb{R}^N)) \cap W_p^1((0, T), L_p(\mathbb{R}^N)) \end{aligned}$$

for $i = 1, \dots, N+1$ and $j = 1, \dots, N-1$, satisfying the compatibility conditions:

$$\begin{aligned} \text{div } \mathbf{u}_{0-} &= f_-|_{t=0} = \text{div } \mathbf{f}_{\text{div}}|_{t=0} && \text{in } \mathbb{R}_-^N, \\ \mu_{*-}D_{iN}(\mathbf{u}_{0-})|_- - \mu_{*+}D_{iN}(\mathbf{u}_{0+})|_+ &= g_i|_{t=0} \quad (i = 1, \dots, N-1) && \text{on } \mathbb{R}_0^N, \\ (\mu_{*+}D_{NN}(\hat{\mathbf{u}}_{0+}) + (\lambda_{*+} - \mu_{*+})\text{div } \hat{\mathbf{u}}_{0+})|_+ &&& \\ &= \sigma_+\Delta' H_0 + g_{N+1}|_{t=0} && \text{on } \mathbb{R}_0^N, \\ u_{0-i}|_- - u_{0+i}|_+ &= h_i|_{t=0} \quad (i = 1, \dots, N-1) && \text{on } \mathbb{R}_0^N. \end{aligned}$$

then, problem (4.4), (4.5), (4.6) admits unique solutions \mathbf{u}_\pm and H with

$$\begin{aligned} \mathbf{u}_\pm &\in L_p((0, T), W_q^2(\mathbb{R}_\pm^N)) \cap W_p^1((0, T), L_q(\mathbb{R}_\pm^N)), \\ H &\in L_p((0, T), W_q^3(\mathbb{R}^N)) \cap W_p^1((0, T), W_q^2(\mathbb{R}^N)) \end{aligned}$$

possessing the estimates:

$$\begin{aligned} &\sum_{\ell=\pm} \{ \|\mathbf{u}_\ell\|_{L_p((0,t), W_q^2(\mathbb{R}_\ell^N))} + \|\partial_t \mathbf{u}_\ell\|_{L_p((0,t), L_q(\mathbb{R}_\ell^N))} \} \\ &\quad + \|\partial_t H\|_{L_p((0,t), W_q^2(\mathbb{R}^N))} + \|H\|_{L_p((0,t), W_q^3(\mathbb{R}^N))} \\ &\leq C e^{\gamma t} \{ \sum_{\ell=\pm} (\|\mathbf{u}_{0\ell}\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_\ell^N)} + \|\mathbf{f}_\ell\|_{L_p((0,t), L_q(\mathbb{R}_\ell^N))}) + \|f_{\text{div}}\|_{L_p((0,t), W_q^1(\mathbb{R}_-^N))} \\ &\quad + \|\mathbf{f}_{\text{div}}\|_{L_p((0,T), L_q(\mathbb{R}_-^N))} + \sum_{i=1}^{N+1} (\|g_i\|_{L_p((0,t), W_q^1(\mathbb{R}^N))} + \|\partial_t g_i\|_{L_p((0,t), W_q^{-1}(\mathbb{R}^N))}) \\ &\quad + \sum_{j=1}^{N-1} (\|h_j\|_{L_p((0,t), W_q^2(\mathbb{R}^N))} + \|\partial_t h_j\|_{L_p((0,t), L_q(\mathbb{R}^N))}) + \|d\|_{L_p((0,t), W_q^2(\mathbb{R}^N))} \} \end{aligned}$$

for any $t \in (0, T)$ with some positive constants C and γ independent of t and T .

And also, we have the following theorem about the maximal L_p - L_q regularity for problem (4.7), (4.8), (4.9).

Theorem 4.2. *Let $1 < p, q < \infty$ and $0 < T < \infty$. Then, for any initial data $\theta_{0\pm} \in B_{q,p}^{2(1-1/p)}(\mathbb{R}_\pm^N)$ and right-hand sides*

$$\tilde{f}_\pm \in L_p((0, T), L_q(\mathbb{R}_\pm^N)), \quad \tilde{g} \in L_p((0, T), W_q^1(\mathbb{R}^N)) \cap W_p^1((0, T), W_q^{-1}(\mathbb{R}^N))$$

satisfying the compatibility condition:

$$[[\theta_0]] = 0, \quad d_{*-} \partial_N \theta_{0-}|_- - d_{*-} \partial_N \theta_{0+}|_+ = \tilde{g}|_{t=0} \quad \text{on } \mathbb{R}_0^N,$$

problem (4.7) and (4.8) admits unique solutions θ_\pm with

$$\theta_\pm \in L_p((0, T), W_q^2(\mathbb{R}_\pm^N)) \cap W_p^1((0, T), L_q(\mathbb{R}_\pm^N))$$

satisfying the estimate:

$$\begin{aligned} & \sum_{\ell=\pm} \{ \|\theta_\ell\|_{L_p((0,t), W_q^2(\mathbb{R}_\ell^N))} + \|\partial_t \theta_\ell\|_{L_p((0,t), L_q(\mathbb{R}_\ell^N))} \} \\ & \leq C^\gamma t \left\{ \sum_{\ell=\pm} (\|\theta_{0\ell}\|_{B_{q,p}^{2(1-1/p)}(\mathbb{R}_\ell^N)} + \|\tilde{f}_\ell\|_{L_p((0,t), L_q(\mathbb{R}_\ell^N)))} \right. \\ & \quad \left. + \|\tilde{g}\|_{L_p((0,t), W_q^1(\mathbb{R}^N))} + \|\partial_t \tilde{g}\|_{L_p((0,t), W_q^{-1}(\mathbb{R}^N))} \right\} \end{aligned}$$

for any $t \in (0, T)$ with some positive constants C and γ independent of t and T .

Remark 3. (1) The proof of Theorem 4.1 is given in [14]. The proof of Theorem 4.2 is found in [3], but it can be proved by using the same argument as in the proof of Theorem 4.1 in [14].

(2) Theorem 3.1 is proved with the help of Theorem 4.1 and Theorem 4.2, the Banach fixed point argument and, some bootstrap arguments. The argument is quite standard, so that we may omit the proof of Theorem 3.1 (cf. Prüss [5]).

5 \mathcal{R} -bounded solution operators

To prove Theorem 4.1, we consider the following generalized resolvent problem:

$$\begin{aligned} \rho_{*+} \lambda \mathbf{u}_+ - \operatorname{Div} \mathbf{S}_{*+}(\mathbf{u}_+) &= \mathbf{f}_+ && \text{in } \mathbb{R}_+^N \\ \rho_{*-} \lambda \mathbf{u}_- - \operatorname{Div} \mathbf{S}_{*-}(\mathbf{u}_-) + \nabla \pi_- &= \mathbf{f}_-, \quad \operatorname{div} \mathbf{u}_- = f_{\operatorname{div}} = \operatorname{div} \mathbf{f}_{\operatorname{div}} && \text{in } \mathbb{R}_-^N \end{aligned} \quad (5.1)$$

subject to the interface condition: for $x \in \mathbb{R}_0^N$

$$\begin{aligned} \mu_{*-} D_{iN}(\mathbf{u}_-)|_- - \mu_{*+} D_{iN}(\mathbf{u}_+)|_+ &= g_i \quad (i = 1, \dots, N-1), \\ (\mu_{*-} D_{NN}(\mathbf{u}_-) - \pi_-)|_- &= \sigma_- \Delta' H + g_N, \\ (\mu_{*+} D_{NN}(\mathbf{u}_+) + (\lambda_{*+} - \mu_{*+}) \operatorname{div} \mathbf{u}_+)|_+ &= \sigma_+ \Delta' H + g_{N+1}, \\ u_{-i}|_- - u_{+i}|_+ &= h_i \quad (i = 1, \dots, N-1), \\ \lambda H - \left(\frac{\rho_{*-}}{\rho_{*-} - \rho_{*+}} u_{-N} - \frac{\rho_{*+}}{\rho_{*-} - \rho_{*+}} u_{+N} \right) &= d, \end{aligned} \quad (5.2)$$

which is corresponding to the time dependent problem (4.4), (4.5), (4.6).

Before stating the main result of this section, we first introduce the definition of \mathcal{R} -boundedness and the operator valued Fourier multiplier theorem due to Weis [19].

Definition 5.1. Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called \mathcal{R} -bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$ -valued random variables on $[0, 1]$ there holds the inequality:

$$\left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j f_j \right\|_Y^p du \right\}^{\frac{1}{p}} \leq C \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(u) f_j \right\|_X^p du \right\}^{\frac{1}{p}}.$$

The smallest such C is called \mathcal{R} -bound of \mathcal{T} , which is denoted by $\mathcal{R}_{\mathcal{L}(X, Y)}(\mathcal{T})$. Here, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y .

Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X valued C^∞ functions having compact supports and the Schwartz space of rapidly decreasing X valued functions, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1, \text{loc}}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \rightarrow \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M \phi = \mathcal{F}^{-1}[M \mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)), \quad (5.3)$$

The following theorem is obtained by Weis [19].

Theorem 5.2. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X, Y)}\left(\left\{\left(\tau \frac{d}{d\tau}\right)^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\right\}\right) \leq \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ . Then, the operator T_M defined in (5.3) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M , we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R}, X), L_p(\mathbb{R}, Y))} \leq C\kappa$$

for some positive constant C depending on p , X and Y .

Remark 4. For the definition of UMD space, we refer to a book due to Amann [1]. For $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W_q^m(\Omega)$ are both UMD spaces.

Theorem 5.3. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Set $\Sigma_{\epsilon, \lambda_0} = \{\lambda \in \Sigma_\epsilon \mid |\lambda| \geq \lambda_0\}$ ($\lambda_0 > 0$) with $\Sigma_\epsilon = \{\lambda = \gamma + i\tau \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}$, and*

$$\begin{aligned} X_q &= \{(\mathbf{f}_+, \mathbf{f}_-, f_{\text{div}}, \mathbf{f}_{\text{div}}, \mathbf{g}, \mathbf{h}, d) \mid \mathbf{f}_+ \in L_q(\mathbb{R}_+^N), \mathbf{f}_-, \mathbf{f}_{\text{div}} \in L_q(\mathbb{R}_-^N), d \in W_q^2(\mathbb{R}^N), \\ &\quad f_{\text{div}} \in W_q^1(\mathbb{R}^N), \mathbf{g} = (g_1, \dots, g_{N+1}) \in W_q^1(\mathbb{R}^N), \mathbf{h} = (h_1, \dots, h_{N-1}) \in W_q^2(\mathbb{R}^N)\}, \\ \mathcal{X}_q &= \{\mathbf{F} = (\mathbf{F}_{+1}, \mathbf{F}_{-1}, F_{-2}, \mathbf{F}_{-3}, \mathbf{F}_{-4}, \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5, F_6) \mid \mathbf{F}_{\pm 1} \in L_q(\mathbb{R}_\pm^N), \\ &\quad F_{-2}, \mathbf{F}_{-3}, \mathbf{F}_{-4} \in L_q(\mathbb{R}_-^N), \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5 \in L_q(\mathbb{R}^N), F_6 \in W_q^2(\mathbb{R}^N)\}. \end{aligned}$$

Then, there exist a constant $\lambda_0 > 0$ and operator families

$$\begin{aligned} \mathcal{A}_\pm(\lambda) &\in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q, W_q^2(\mathbb{R}_\pm^N))), \quad \mathcal{P}_- \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q, \hat{W}_q^1(\mathbb{R}_-^N))), \\ \mathcal{H}(\lambda) &\in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X_q, W_q^3(\mathbb{R}^N))) \end{aligned}$$

such that $\mathbf{u}_\pm = \mathcal{A}_\pm(\lambda)\mathbf{F}_\lambda$, $\pi_- = \mathcal{P}_-(\lambda)\mathbf{F}_\lambda$ and $H = \mathcal{H}(\lambda)\mathbf{F}_\lambda$ are unique solutions of problem (5.1) and (5.2) for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$ and $\mathbf{F} = (\mathbf{f}_+, \mathbf{f}_-, f_{\text{div}}, \mathbf{f}_{\text{div}}, \mathbf{g}, \mathbf{h}, d) \in X_q$, and we have

$$\begin{aligned}\mathcal{R}_{\mathcal{L}(\mathcal{X}_q, L_q(\mathbb{R}_\pm^N))}(\{(\tau\partial_\tau)^\ell G_\lambda^1 \mathcal{A}_\pm(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq c \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q, L_q(\mathbb{R}_\pm^N))}(\{(\tau\partial_\tau)^\ell \nabla \mathcal{P}_-(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq c \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q, W_q^2(\mathbb{R}_\pm^N))}(\{(\tau\partial_\tau)^\ell G_\lambda^2 \mathcal{H}(\lambda) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq c \quad (\ell = 0, 1)\end{aligned}$$

with some constant c . Here,

$$\begin{aligned}G_\lambda^1 \mathcal{A}_\pm(\lambda) &= (\lambda \mathcal{A}_\pm(\lambda), \lambda^{1/2} \nabla \mathcal{A}_\pm(\lambda), \nabla^2 \mathcal{A}_\pm(\lambda)), \quad G_\lambda^2 \mathcal{H}(\lambda) = (\lambda \mathcal{H}(\lambda), \nabla \mathcal{H}(\lambda)), \\ \mathbf{F}_\lambda &= (\mathbf{f}_+, \mathbf{f}_-, \lambda^{1/2} f_{\text{div}}, \nabla f_{\text{div}}, \lambda \mathbf{f}_{\text{div}}, \lambda^{1/2} \mathbf{g}, \nabla \mathbf{g}, \lambda \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \nabla^2 \mathbf{h}, d), \\ \hat{W}_q^1(\mathbb{R}^N) &= \{\pi_- \in L_{q, \text{loc}}(\mathbb{R}^N) \mid \nabla \pi_- \in L_q(\mathbb{R}^N)\},\end{aligned}$$

$\text{Hol}(U, X)$ denotes the set of all holomorphic functions defined on U with their values in X , $\nabla = (\partial_1, \dots, \partial_N)$ and $\nabla^2 = (\partial_i \partial_j \mid i, j = 1, \dots, N)$.

Remark 5. (1) $\mathbf{F}_{\pm 1}, F_{-2}, \mathbf{F}_{-3}, \mathbf{F}_{-4}, \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5$ and F_6 are corresponding variables to $\mathbf{f}_\pm, \lambda^{1/2} f_{\text{div}}, \nabla f_{\text{div}}, \mathbf{f}_{\text{div}}, \lambda^{1/2} \mathbf{g}, \nabla \mathbf{g}, \lambda \mathbf{h}, \lambda^{1/2} \nabla \mathbf{h}, \nabla^2 \mathbf{h}$ and d , respectively.

(2) The proof of Theorem 5.3 is given in [14].

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